Geometric analysis - Yamabe problem and Geometric flow

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Abstract

These are lecture notes that I typed up for Professor Pak Tung Ho's course (MAT6320) on Geometric analysis in Spring 2018. I should note that these notes are not polished and hence might be riddled with errors. If you notice any typos or errors, please do contact me at willkwon@sogang.ac.kr

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Introduction to Yamabe problem

1.1 Introduction

We discuss the motivation of Yamabe problem. Let M be a C^∞-manifold and g_0 be a Riemannian metric on M. We have

Exercise 1.1. Let u be a positive smooth function on M. Define $g = ug_0$. Show that g is a Riemannian metric on M.

Definition 1.2. We say that g is *conformal* to g_0 if $g = ug_0$ for some $u > 0$, $u \in C^\infty(M)$.

Exercise 1.3. Show that the relation "is conformal to" is an equivalence relation.

Definition 1.4. Given a Riemannian metric g_0, the equivalence class $[g_0]$ is called *conformal class* of g_0, i.e.,

$$[g_0] = \{\text{Riemannain metric } g : g \text{ is conformal to } g_0\}.$$

Question. Given a C^∞-manifold M, how many conformal classes are there?

The following theorem is a classical.

Theorem 1.5 (Uniformization theorem)**.** *Suppose M is a compact 2-dimensional manifold and g_0 is a Riemannian metric on M. Then there exists a $g \in [g_0]$ such that the Gaussian curvature K_g of g is constant.*

Most of people are not familar with this type of theorem. Assuming the following fact,

Fact. If (M, g) is a compact 2-dimensional manifold which has constant Gaussian curvature K_g, then (M, g) is one of the following: (genus).

then we obtain the familiar one. So we want to generalize the uniformization theorem to higher dimension.

Problem (Yamabe problem)**.** Let M be an n-dimensional compact manifold g_0 a Riemannian metric on M, where $n \geq 3$. Find $g \in [g_0]$ such that the scalar curatuve of g is constant.

Remark. For your information, in dimension 2, $K_g = 2R_g$, $\mathrm{Ric}_g = \frac{1}{2}K_g g$. If M is not compact, then the problem may not solvable.

The Yamabe problem can be stated as a PDE problem.

Proposition 1.6. *Let M be an n-dimensional compact manifold, where $n \geq 3$. If $g = u^{\frac{4}{n-2}} g_0$ for some $0 < u \in C^\infty(M)$, then*

$$-\frac{4(n-1)}{n-2}\triangle_{g_0} u + R_{g_0} u = R_g u^{\frac{n+2}{n-2}}. \tag{1.1}$$

Here $\triangle_{g_0}$ is a Laplacian of g_0, R_{g_0} a scalar curvature of g_0, R_g a scalar curvature of g, respectively.

Proof. The proof is quite tedious but you have to do it one time. To prove this, we do this in two steps:

1. we write down the Christoffel symbols of g in terms of the Christoffel symbols of g_0.

2. we write down the curvature of g in terms of the Christoffel symbols of g_0.

For convenience, write $\widetilde{g} = ug$, where u is a positive smooth function on M and let $\{X_i\}$ be the normal frame at $p \in M$ with respect to g, i.e., $g(X_i, X_j) = \delta_{ij}$ in a neighborhood of p so that $\Gamma_{ij}^k = 0$ at p.[1] So the Christoffel symbol of $\widetilde{g}$ is given by

$$\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \frac{1}{2}\widetilde{g}^{kl}\left(\frac{\partial}{\partial x_i}\widetilde{g}_{jl} + \frac{\partial}{\partial x_j}\widetilde{g}_{il} - \frac{\partial}{\partial x_l}\widetilde{g}_{ij}\right) \\
&= \frac{1}{2}u^{-1}g^{kl}\left[\frac{\partial}{\partial x_i}(ug_{jl}) + \frac{\partial}{\partial x_j}(ug_{il}) - \frac{\partial}{\partial x_l}(ug_{ij})\right] \\
&= \frac{1}{2}u^{-1}g^{kl}u\left[\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right] + \frac{1}{2}u^{-1}g^{kl}\left[g_{jl}\frac{\partial}{\partial x_i}u + g_{il}\frac{\partial}{\partial x_j}u - g_{ij}\frac{\partial}{\partial x_l}u\right] \\
&= \Gamma_{ij}^k + \frac{1}{2}\left(\delta_{jk}\frac{\partial}{\partial x_i}\log u + \delta_{ik}\frac{\partial}{\partial x_j}\log u - \delta_{ij}\frac{\partial}{\partial x_k}\log u\right).
\end{aligned}$$

The last part comes from the assumption on normal frame. So we have

$$\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \Gamma_{ij}^k \qquad \text{if } i, j, k \text{ are all distinct} \\
\widetilde{\Gamma}_{ii}^k &= \Gamma_{ii}^k - \frac{1}{2}\frac{\partial}{\partial x_k}\log u \\
\widetilde{\Gamma}_{ij}^i &= \Gamma_{ij}^i + \frac{1}{2}\frac{\partial}{\partial x_j}\log u = \widetilde{\Gamma}_{ji}^i \\
\widetilde{\Gamma}_{ii}^i &= \Gamma_{ii}^i + \frac{1}{2}\frac{\partial}{\partial x_i}\log u.
\end{aligned} \tag{1.2}$$

The curvature tensor of $\widetilde{g}$ is given by

$$\widetilde{R}_{ijk}^s = \widetilde{\Gamma}_{ik}^l\widetilde{\Gamma}_{jl}^s - \widetilde{\Gamma}_{jk}^l\widetilde{\Gamma}_{il}^s + \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{ik}^s - \frac{\partial}{\partial x_i}\widetilde{\Gamma}_{jk}^s,$$

and the curvature tensor of g is given by

$$R_{ijk}^s = \Gamma_{ik}^l\Gamma_{jl}^s - \Gamma_{jk}^l\Gamma_{il}^s + \frac{\partial}{\partial x_j}\Gamma_{ik}^s - \frac{\partial}{\partial x_i}\Gamma_{jk}^s.$$

In particular, at a point p,

$$R_{ijk}^s = \frac{\partial}{\partial x_j}\Gamma_{ik}^s - \frac{\partial}{\partial x_i}\Gamma_{jk}^s.$$

Now the sectional curvature of $\widetilde{g}$ in $\{X_i, X_j\}$ at p is given by

$$\begin{aligned}
\widetilde{K}(X_i, X_j) &= \frac{\widetilde{R}_{iji}^s\widetilde{g}_{sj}}{\widetilde{g}_{ii}\widetilde{g}_{jj} - \widetilde{g}_{ij}^2} \\
&= \frac{\widetilde{R}_{iji}^s u g_{sj}}{(ug_{ii})(ug_{jj}) - (ug_{ij})^2} \\
&= u^{-1}\widetilde{R}_{iji}^j.
\end{aligned}$$

This implies

$$u\widetilde{K}(X_iX_j) = \widetilde{R}_{iji}^j$$

[1] Be careful! The Christoffel symbol vanishes only at p, not on the neighborhood of p!

$$= \widetilde{\Gamma}_{ii}^{l}\widetilde{\Gamma}_{jl}^{j} - \widetilde{\Gamma}_{ji}^{l}\widetilde{\Gamma}_{il}^{j} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{ii}^{j} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}_{ji}^{j}.$$

Note that

$$\sum_{l=1,l\neq i\neq j}^{n} \widetilde{\Gamma}_{ji}^{l}\widetilde{\Gamma}_{il}^{j} = \sum_{l=1,l\neq i\neq j}^{n} \Gamma_{ij}^{l}\Gamma_{il}^{j} = 0$$

since $l \neq i \neq j$ and $\Gamma_{ij}^{l} = 0$ at p.

Now from (1.2), we have

$$\begin{aligned} u\widetilde{K}(X_i, X_j) &= \sum_{l=1,l\neq i}^{n} \widetilde{\Gamma}_{ii}^{l}\widetilde{\Gamma}_{jl}^{j} + \widetilde{\Gamma}_{ii}^{i}\widetilde{\Gamma}_{ji}^{j} \qquad (1.3)\\ &\quad - \left(\widetilde{\Gamma}_{ji}^{j}\widetilde{\Gamma}_{ij}^{j} + \widetilde{\Gamma}_{ji}^{i}\widetilde{\Gamma}_{ii}^{j}\right) + \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{ii}^{j} - \frac{\partial}{\partial x_j}\widetilde{\Gamma}_{ji}^{j}\\ &= \sum_{l=1,l\neq i}^{n} \left(-\frac{1}{2}(\log u)_l\right)\left(\frac{1}{2}(\log u)_l\right) + \left(\frac{1}{2}(\log u)_i\right)\left(\frac{1}{2}(\log u)_i\right)\\ &\quad - \left(\frac{1}{2}(\log u)_i\right)^2 - \left(\frac{1}{2}(\log u)_j\right)\left(-\frac{1}{2}(\log u)_j\right)\\ &\quad + \frac{\partial}{\partial x_j}\left(\Gamma_{ii}^{j} - \frac{1}{2}(\log u)_j\right) - \frac{\partial}{\partial x_i}\left(\Gamma_{ji}^{j} + \frac{1}{2}(\log u)_i\right)\\ &= -\frac{1}{4}\sum_{l=1}^{n}\left[(\log u)_l\right]^2 + \frac{1}{4}\left[(\log u)_j\right]^2 + \frac{1}{4}\left[(\log u)_i\right]^2\\ &\quad - \frac{1}{2}(\log u)_{jj} - \frac{1}{2}(\log u)_{ii} + K(X_i, X_j). \end{aligned}$$

So we can compute the Ricci curvature of $\widetilde{g}$ in terms of Ricci curvature of g. Also, we can compute the scalar curvature of $\widetilde{g}$ in terms of the scalar curvature of g. So

$$\begin{aligned} uR_{\widetilde{g}} &= \sum_{i\neq j} u\widetilde{K}(X_i, X_j) \qquad (1.4)\\ &= \sum_{i\neq j} -\frac{1}{4}\sum_{l=1}^{n}\left[(\log u)_l\right]^2 + \frac{1}{4}\left[(\log u)_j\right]^2 + \frac{1}{4}\left[(\log u)_i\right]^2\\ &\quad - \frac{1}{2}(\log u)_{jj} - \frac{1}{2}(\log u)_{ii} + K(X_i, X_j)\\ &= -\frac{1}{4}\left(n^2 - n\right)\sum_{l=1}^{n}\left[(\log u)_l\right]^2 + \frac{1}{2}(n-1)\sum_{i=1}^{n}\left[(\log u)_i\right]^2 - (n-1)\sum_{i=1}^{n}(\log u)_{ii} + R_g. \end{aligned}$$

In normal coordinate, we see that

$$\nabla_g(\log u) = \nabla(\log u), \quad \triangle_g u = \triangle u.$$

Thus, we get

$$uR_{\widetilde{g}} = -\frac{1}{4}(n-2)(n-1)\left|\nabla_g(\log u)\right|^2 - (n-1)\triangle_g(\log u) + R_g.$$

If $n = 2$, then

$$uR_{\widetilde{g}} = R_g - \triangle_g(\log u),$$

when $\widetilde{g} = ug$. Write $u = e^v$. Then

$$e^v R_{\widetilde{g}} = R_g - \triangle_g v$$

so that

$$e^v \frac{1}{2} K_{\widetilde{g}} = \frac{1}{2} K_g - \triangle_g v,$$

where K_g is a Gaussian curvature.

If $n \geq 3$, then we let $u = v^{\frac{4}{n-2}}$, i.e., $\widetilde{g} = v^{\frac{4}{n-2}} g$, then

$$\begin{aligned} \log u &= \frac{4}{n-2} \log v \\ \nabla_g (\log u) &= \frac{4}{n-2} \nabla_g (\log v) = \frac{4}{n-2} \frac{\nabla_g v}{v} \\ \triangle_g (\log u) &= -\frac{4}{n-2} \frac{|\nabla_g v|^2}{v^2} + \frac{4}{n-2} \frac{\triangle_g v}{v}. \end{aligned}$$

Put these this into (1.4), we obtain

$$v^{\frac{4}{n-2}} R_{\widetilde{g}} = R_g - \frac{4(n-1)}{n-2} \frac{\triangle_g v}{v},$$

which is (1.1). □

Remark. (i) The reason to take $u = e^v$ if $n = 2$, and $u = v^{\frac{4}{n-2}}$ when $n \geq 3$ is to cancel the gradient term.

(ii) The Yamabe problem is equivalent to find a positive solution $u > 0$, $u \in C^\infty(M)$ of (1.1) with R_g being constant.

(iii) The equation (1.1) is called *Yamabe PDE*, which is a second order semilinear elliptic equation.

To solve (1.1), we define some concepts.

Definition 1.7. Let M be an n-dimensional compact manifold, where $n \geq 3$ and let g_0 be a Riemannian metric on M. For any $u > 0$, $u \in C^\infty(M)$, we can define

$$E_{g_0}(u) = \frac{\int_M \frac{4(n-1)}{n-2} |\nabla_{g_0} u|^2 + R_{g_0} u^2 dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}},$$

the *Yamabe energy*.

If the metric is ambient, we drop g_0 in the notation.

Here $\nabla_{g_0} u$ is a vector field such that

$$\langle X, \nabla_{g_0} u \rangle_{g_0} = X(u) \quad \text{for any other vector field } X.$$

Remark. Multiply u to (1.1) and take integration by part. Then we obtain each part of the integrand.

Let us state some properties of Yamabe energy.

(1) The Yamabe energy is scalar invariant in this sense:

$$E(cu) = E(u)$$

for any constant $c \neq 0$.

(2) If $u>0$ and $g=u^{\frac{4}{n-2}}g_0$, then

$$dV_g = u^{\frac{2n}{n-2}} dV_{g_0}. \tag{1.5}$$

Indeed, we have

$$\begin{aligned} dV_g &= \sqrt{\det(g^{ij})}dx_1\wedge\cdots\wedge dx_n = \sqrt{\det(u^{\frac{4}{n-2}}g_0)}dx_1\wedge\cdots\wedge dx_n \\ &= \sqrt{u^{\frac{4n}{n-2}}\det g_0}dx_1\wedge\cdots\wedge dx_n \\ &= u^{\frac{2n}{n-2}}\sqrt{\det g_0}dx_1\wedge\cdots\wedge dx_n \\ &= u^{\frac{2n}{n-2}}dV_{g_0}. \end{aligned}$$

(3) Based on (2), if $g=u^{\frac{4}{n-2}}g_0$ and u is a solution of (1.1), then

$$\int_M u^{\frac{2n}{n-2}}dV_{g_0} = \mathrm{Vol}\,(M,g) \tag{1.6}$$

and

$$\begin{aligned} &\int_M \left(\frac{4\,(n-1)}{n-2}\left|\nabla_{g_0}u\right|^2 + R_{g_0}u^2\right)dV_{g_0} \\ &= \int_M \left(-\frac{4\,(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u\right)udV_{g_0} \\ &= \int_M R_g u^{\frac{2n}{n-2}}dV_{g_0} = \int_M R_g dV_g. \end{aligned}$$

Thus

$$E\,(u) = \frac{\int_M R_g dV_g}{\mathrm{Vol}\,(M,g)^{\frac{n-2}{n}}},$$

where $g=u^{\frac{4}{n-2}}g_0$. We call $\int_M R_g dV_g$, the *total scalar curvature*.

Proposition 1.8. *If u is a critical point of the Yamabe energy, then u satisfies the Yamabe equation with R_g being constant.*

Proof. If u is a crticial point of E, then

$$\begin{aligned} 0 &= \frac{d}{dt}E\,(u+t\varphi)\;|_{t=0}, \\ &= \frac{d}{dt}\frac{\int_M\left(\frac{4(n-1)}{n-2}\left|\nabla_{g_0}u+t\nabla_{g_0}\varphi\right|^2 + R_{g_0}\,(u+t\varphi)^2\right)dV_{g_0}}{\left(\int_M (u+t\varphi)^{\frac{2n}{n-2}}\,dV_{g_0}\right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M \frac{4(n-1)}{n-2}2\left\langle\nabla_{g_0}u,\nabla_{g_0}\varphi\right\rangle + 2R_{g_0}u\varphi dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} \\ &\quad -\frac{n-2}{n}\frac{\int_M u^{\frac{2n}{n-2}-1}\frac{2n}{n-2}\varphi dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}}dV_{g_0}\right)^{\frac{n-2}{n}+1}}\left(\int_M\left(\frac{4\,(n-1)}{n-2}\left|\nabla_{g_0}u\right|^2 + R_{g_0}u^2\right)dV_{g_0}\right) \\ &= \frac{2\int_M\left(-\frac{4(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u\right)\varphi dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}}dV_{g_0}\right)^{\frac{n-2}{n}}} - \frac{2E\,(u)\int_M u^{\frac{n+2}{n-2}}\varphi dV_{g_0}}{\int_M u^{\frac{2n}{n-2}}dV_{g_0}} \end{aligned}$$

$$= \frac{2\int_M \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u - \frac{E(u)}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{2}{n}}} u^{\frac{n+2}{n-2}} \right) \varphi dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}}$$

holds for any $\varphi \in C^\infty(M)$. Thus,

$$-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u = \frac{E(u)}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{2}{n}}} u^{\frac{n+2}{n-2}},$$

i.e., for $g = u^{\frac{4}{n-2}} g_0$ which has constant scalar curvature, i.e.,

$$R_g = \frac{E(u)}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{2}{n}}}. \qquad \square$$

Thus by Proposition 1.8, we reduced the Yamabe problem to find a critical point of Yamabe energy.

1.2 Yamabe constant and the conformal Laplacian

Given a compact Riemannian manifold (M, g), the Yamabe energy of a function $0 < u \in C^\infty(M)$ is given by

$$E(u) = \frac{\int \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}}.$$

If M is not compact, then we must restrict u which has a compact support. We will see it later.

Definition 1.9. The *Yamabe constant* of (M, g) is given by

$$Y(M, g) = \inf \{E(u) : 0 < u \in C^\infty(M)\}.$$

Proposition 1.10. *$Y(M, g)$ is a finite number.*

Proof. If $u = 1$, then $Y(M, g) \leq E(1) < \infty$. If $\min_M R_g \geq 0$, then by definition of the Yamabe energy, $E(u) \geq 0$. So $Y(M, g) \geq 0$ in this case. If $\min_M R_g < 0$, then by Hölder's inequality

$$\begin{aligned}
\int_M R_g u^2 dV_g &\geq \left(\min_M R_g\right) \int_M u^2 dV_g \\
&\geq \left(\min_M R_g\right) \left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} \left(\int_M dV_g\right)^{\frac{2}{n}}.
\end{aligned}$$

So

$$\begin{aligned}
E(u) &\geq \frac{(\min_M R_g) \left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} \left(\int_M dV_g\right)^{\frac{2}{n}}}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \\
&= \left(\min_M R_g\right) \left(\int_M dV_g\right)^{\frac{2}{n}}.
\end{aligned}$$

This completes the proof. $\square$

Definition 1.11. Given a Riemannian manifold (M, g), the *conformal Laplacian* L_g is defined by

$$L_g = -\frac{4\,(n-1)}{n-2}\triangle_g + R_g.$$

Clearly, $L_g : C^\infty(M) \to C^\infty(M)$. So we can rewrite (1.1) as

$$L_{g_0} u = R_g u^{\frac{n+2}{n-2}},$$

where $g = u^{\frac{4}{n-2}} g_0$.

The conformal Laplacian has some properties[2].

Proposition 1.12. *For any $\varphi \in C^\infty(M)$, we have*

$$u^{-\frac{n+2}{n-2}} L_{g_0}(u\varphi) = L_{u^{\frac{4}{n-2}} g_0}(\varphi), \tag{1.7}$$

where $0 < u \in C^\infty(M)$, i.e., the conformal Laplacian is conformal invariant in this sense.

Proof. Fix $u, \varphi > 0$, $\varphi \in C^\infty(M)$. Then using Yamabe equation, we have

$$\begin{aligned} L_{g_0}(u\varphi) &= R_{(u\varphi)^{\frac{4}{n-2}} g_0} (u\varphi)^{\frac{n+2}{n-2}} \\ &= u^{\frac{n+2}{n-2}} R_{(u\varphi)^{\frac{4}{n-2}} g_0} \varphi^{\frac{n+2}{n-2}} \\ &= u^{\frac{n+2}{n-2}} R_{\varphi^{\frac{4}{n-2}}\left(u^{\frac{4}{n-2}} g_0\right)} \varphi^{\frac{n+2}{n-2}} \\ &= u^{\frac{n+2}{n-2}} L_{u^{\frac{4}{n-2}} g_0}(\varphi). \end{aligned}$$

□

This proves (1.7) under the assumption $\varphi > 0$. Then applying the limiting argument, (1.7) holds when $\varphi \geq 0$.

For general φ, decompose $\varphi = \varphi^+ - \varphi^-$ as usual. Although φ^+ and φ^- is not smooth, we take a mollifier, i.e., $\varphi_\varepsilon^+ = (\varphi^+ * \rho_\varepsilon)$, where $\{\rho_\varepsilon\}$ is a nonnegative mollifier. Then φ_ε^+ is smooth and nonnegative and φ_ε^+ converges to φ^+ pointwise. Similarly, $\varphi_\varepsilon^- \to \varphi^-$ pointwise and so $\varphi_\varepsilon^+ - \varphi_\varepsilon^- \to \varphi$ pointwise.

Now by linearlity of conformal Laplacian, we have

$$L_{u^{\frac{4}{n-2}} g_0}(\varphi_\varepsilon^+ - \varphi_\varepsilon^-) = u^{-\frac{n+2}{n-2}} L_{g_0}(u(\varphi_\varepsilon^+ - \varphi_\varepsilon^-))$$

Now observe that $\varphi_\varepsilon^+ - \varphi_\varepsilon^- = \varphi_\varepsilon$ and it converges φ pointwise and uniformly. Now let $\varepsilon \to 0$. This completes the proof.

Remark. (i) One can prove that L_{g_0} is continuous on $W^{1,2}(M)$. Then applying the density argument, (1.7) holds for $W^{1,2}(M)$.

(ii) One can also show it directly without using Yamabe equation.

So the Yamabe energy is written as

$$E(u) = \frac{\int_M \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}}$$

[2] Not just for the writing it in LaTeX easily

$$= \frac{\int_M uL_g u dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}}.$$

As one can see, Yamabe energy depends on the background metric. But, in fact, we have

Lemma 1.13. *The Yamabe constant of* (M, g_0) *depends only on* $[g_0]$*, i.e., if* $g \in [g_0]$*,*

$$Y(M, g) = Y(M, g_0).$$

Proof. Since $g \in [g_0]$, we can wrtie $g = \varphi^{\frac{4}{n-2}} g_0$. Then by Proposition 1.12 and (1.5), we have

$$\begin{aligned} E_g(u) &= \frac{\int_M uL_g u dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M uL_{\varphi^{\frac{4}{n-2}} g_0} u dV_{\varphi^{\frac{4}{n-2}} g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{\varphi^{\frac{4}{n-2}} g_0}\right)^{\frac{n-2}{n}}} \\ &= \frac{\int_M u\varphi^{-\frac{n+2}{n-2}} L_{g_0}(u\varphi) \varphi^{\frac{2n}{n-2}} dV_{g_0}}{\left(\int_M (u\varphi)^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}} \\ &= E_{g_0}(u\varphi). \end{aligned}$$

Thus,

$$\begin{aligned} Y(M, g_0) &= \inf\left\{E_{g_0}(u) : 0 < u \in C^\infty(M)\right\} \\ &= \inf\left\{E_{g_0}(\varphi u) : 0 < u \in C^\infty(M)\right\} \\ &= \inf\left\{E_g(u) : 0 < u \in C^\infty(M)\right\} \\ &= Y(M, g). \end{aligned}$$

So we are done. □

Remark. (i) In view of Lemma 1.13, the Yamabe constant of (M, g) is sometimes written as $Y(M, [g])$.

(ii) If g, g_0 are Riemannian metrics in M such that g and g_0 are not conformal, then $Y(M, [g_0])$ and $Y(M, [g])$ may not equal.

(iii) One can propose the following problem. It is actually an open problem: given a compact smooth manifold, what is

$$\{Y(M, g) : g \text{ is Riemannian metric on } M\}?$$

Some result is available when M is a 3-dimensional real projective plane, but essentially there is no techinque to attack this kind of problem.

Define

$$E(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2\right) dV_g}{\left(\int_M |u|^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}}.$$

Suppose $u \neq 0$ in $W^{1,2}(M)$. Then by the Sobolev embedding theorem, $u \in L^{\frac{2n}{n-2}}(M)$ and so $E(u)$ is well-defined. Note that for $u \in W^{1,2}(M)$, $|\nabla_g |u|| = |\nabla_g u|$ a.e. So $E(u) = E(|u|)$.

It is useful to observe that the restriction to smooth positive functions in the definition of Yamabe constant $Y(M, g)$ is unnecessary.

Lemma 1.14. *We have*

$$Y(M,g) = \inf\left\{E(u) : 0 \not\equiv u \in W^{1,2}(M)\right\}.$$

Proof. Following the proof of Proposition 1.10, we see that $\inf\left\{E(u) : 0 \not\equiv u \in W^{1,2}(M)\right\}$ is a finite number. Let us denote $y = \inf\left\{E(u) : 0 \not\equiv u \in W^{1,2}(M)\right\}$. Clearly, $Y(M,g) \geq y$.

Now let $\varepsilon > 0$ be given. Then there exists $u \in W^{1,2}(M)$ such that $u \neq 0$ and $E(u) < y + \varepsilon$. Since $E(u) = E(|u|)$, we may assume $u \geq 0$ but not identically zero. Since M is compact, $C^\infty(M)$ is dense in $W^{1,2}(M)$. So there exists a sequence $u_k \in C^\infty(M)$ satisfying $u_k > 0$ and $u_k \to u$ in $W^{1,2}(M)$.[3] Also, by Sobolev embedding theorem, $u_k \to u$ in $L^{\frac{2n}{n-2}}(M)$. Thus, $\lim_{k\to\infty} E(u_k) = E(u)$. Since

$$Y(M,g) \leq E(u_k)$$

for all k, letting $k \to \infty$, we have

$$Y(M,g) \leq E(u) < y + \varepsilon,$$

i.e., $Y(M,g) \leq \inf\left\{E(u) : 0 \not\equiv u \in W^{1,2}(M)\right\} + \varepsilon$. The proof is completed by letting $\varepsilon \to 0$. □

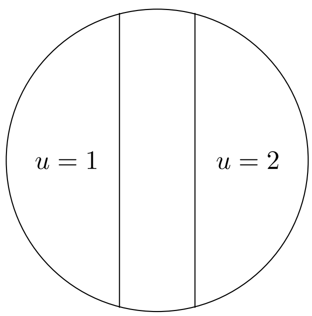
Remark. Note that

$$\sup\left\{E(u) : 0 < u \in C^\infty(M)\right\} = \infty.$$

Choose a small part U of M so that $\mathrm{Vol}(U,g) = \varepsilon$. Define $u \equiv 1$ on left part of U, Define $u \equiv 2$ on the other side of U. Then $|\nabla_g u| \approx \frac{C}{\varepsilon}$ in U. So

$$\begin{aligned} E(u) &= \frac{\int_M \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)} \geq \frac{\frac{4(n-1)}{n-2}\int_M |\nabla_g u|^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} - C \\ &\geq \frac{\frac{4(n-1)}{n-2}\int_U \frac{C}{\varepsilon^2} dV_g}{\left(\int_{M\setminus U} 2^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} - C \\ &\geq \frac{C}{\varepsilon^2}\frac{\mathrm{Vol}(U,g)}{\mathrm{Vol}(M,g) - \varepsilon} - C \to \infty \end{aligned}$$

as $\varepsilon \to 0+$.



Recall our strategy of the Yamabe equation. From Proposition 1.8, it suffices to find a critical point of Yamabe energy. One classical approach is to find a minimizer of the energy. There are another strategy for solving Yamabe problem since we just need to find a critical point of the energy. Using geometric flow approach, we can also find a critical point, which will see it later.

[3] Is it okay?

Note that the conformal Laplacian is a second-order elliptic differential operator. Let $\lambda_1(g)$ be the first eigenvalue of the conformal Laplacian, i.e., $L_g f = \lambda_1(g) f$ for some $f \in C^\infty(M)$. By Courant nodal theorem, f does not change sign. By replacing f by $-f$, we can assume that $f > 0$. By Raleigh quotient,

$$\lambda_1(g) = \inf\left\{\frac{\int_M uL_g u dV_g}{\int_M u^2 dV_g} : 0 \neq u \in C^\infty(M)\right\}. \tag{1.8}$$

Remark. For $n \times n$ matrix A,

$$\lambda_1(A) = \inf\left\{\frac{\langle Ax, x\rangle}{\langle x, x\rangle} : x \neq 0\right\}.$$

Proposition 1.15. *$\lambda_1(g)$ and $Y(M, g)$ have the same sign. This implies the sign of $\lambda_1(g)$ is a conformal invariant.*

Remark. $\lambda_1(g)$ is not conformal invariant. (Hint: for a constant $c > 0$, show that $\lambda_1(cg_0) = c^{-1}\lambda_1(g_0)$)

Proof of Proposition 1.15. First, we show $\lambda_1(g) < 0$ if and only if $Y(M, g) < 0$. Assume $\lambda_1(g) < 0$. Then there exists $f \in C^\infty(M)$ which is positive such that $L_g f = \lambda_1(g) f$ with $\lambda_1(g) < 0$.

Now note that

$$\int_M fL_g f dV_g = \lambda_1(g) \int f^2 dV_g.$$

Since

$$E(f) = \frac{\int_M fL_g f dV_g}{\left(\int_M f^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} = \frac{\lambda_1(g) \int f^2 dV_g}{\left(\int_M f^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} < 0$$

since $\lambda_1(g) < 0$. So

$$Y(M, g) \leq E(f) < 0.$$

Similarly, we see that $\lambda_1(g) \geq 0$ implies $Y(M, g) \geq 0$.

Suppose $Y(M, g) < 0$. Then there exists a positive function $u \in C^\infty(M)$ such that $E(u) < 0$. So

$$\int_M uL_g u dV_g < 0.$$

This implies $\lambda_1(g) < 0$ by (1.8).

Suppose $Y(M, g) \geq 0$. Given any $0 \neq f \in C^\infty(M)$,

$$Y(M, g) \leq E(f).$$

So by Hölder's inequality,

$$\begin{aligned}\frac{\int_M fL_g f dV_g}{\int_M f^2 dV_g} &= \frac{\int_M fL_g f dV_g}{\left(\int_M f^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \frac{\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}{\int_M f^2 dV_g} \\ &\geq c(M, n) Y(M, g)\end{aligned}$$

for some positive constant c depending on M and n. Hence by taking infimum, we obtain the desired result. ☐

Proposition 1.16. *If $Y(M,g) < 0$, then there exists $\tilde{g} \in [g]$ with $R_{\tilde{g}} < 0$. That is to say, there exists a smooth function $0 < u \in C^{\infty}(M)$ such that $\tilde{g} = u^{\frac{4}{n-2}} g$ has negative scalar curvature.*

Proof. By Proposition 1.15, there exists a $0 < f \in C^{\infty}(M)$, $L_g f = \lambda_1(g) f$ with $\lambda_1(g) < 0$. Now consider $\tilde{g} = f^{\frac{4}{n-2}} g$. Then from Yamabe equation and definition of Raleigh quotient, we have

$$\begin{aligned} R_{\tilde{g}} &= f^{-\frac{n+2}{n-2}} L_g f \\ &= f^{-\frac{n+2}{n-2}} \lambda_1(g) f < 0, \end{aligned}$$

which completes the proof. □

Remark. This proposition does not imply that scalar curvature is constant.

By the same proof, we can also show:

Proposition 1.17. *If $Y(M,g) \geq 0$, then there exists $\tilde{g} \in [g]$ with $R_{\tilde{g}} \geq 0$.*

We will see that the situation is very different when $Y(M,g) \geq 0$ and $Y(M,g) < 0$.
In fact, the converse is also true.

Lemma 1.18. *(i) If there exists $g \in [g_0]$ such that $R_g < 0$, then $Y(M,g_0) < 0$.*
(ii) If there exists $g \in [g_0]$ such that $R_g \geq 0$, then $Y(M,g_0) \geq 0$.

Proof. (i) If $g \in [g_0]$ such that $R_g < 0$, then

$$E_g(1) = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}} < 0$$

and thus $Y(M,g) \leq E_g(1) < 0$. Since $Y(M,g)$ is conformal invariant, $Y(M,g_0) < 0$.
(ii) If $g \in [g_0]$ such that $R_g \geq 0$,

$$E_g(u) = \frac{\int_M \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \geq 0$$

for all $0 < u \in C^{\infty}(M)$. Now taking infimum, we obtain

$$Y(M,g) = \inf\{E_g(u) : 0 < u \in C^{\infty}(M)\} = Y(M,g_0).$$ □

This easy lemma helps us to compute the Yamabe constant for some manifolds exactly.

Lemma 1.19. *Let $\mathbb{T}^n$ be the n-dimensional torus, i.e., $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let g_0 be a flat metric on $\mathbb{T}^n$. Then $Y(M,g_0) = 0$.*

Proof. Since g_0 is flat,

$$E_{g_0}(1) = \frac{\int_M R_{g_0} dV_{g_0}}{\left(\int_M dV_{g_0}\right)^{\frac{n-2}{n}}} = 0.$$

So $Y(M,g_0) \leq E_{g_0}(1) = 0$.
We claim that there is no $g \in [g_0]$ such that $R_g < 0$. Suppose not. Then there exists $g \in [g_0]$ such that $R_g < 0$. Then multiply u and taking integration by parts, we get

$$\frac{4(n-1)}{n-2} \int_M |\nabla_{g_0} u|^2 dV_{g_0} = \int_M R_g u^{\frac{n+2}{n-2}} dV_{g_0} < 0,$$

a contradiction. Thus, $Y(M,g_0) \geq 0$. This completes the proof. □

In fact, computing the Yamabe constant of a Riemannian manifold (M, g_0) is extremely difficult.

(i) For example, it is difficult to compute $\left(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \times g_{\mathbb{S}^2}\right)$ since the energy behaves badly for product metric. For the product metric $g_M \times g_N$ on $M \times N$, the energy

$$E(u) = \frac{\int_{M\times N} \frac{4(m+n-1)}{m+n-2} \left|\nabla_{g_M\times g_N} u\right|^2 + R_{g_M\times g_N} u^2 dV_{g_M\times g_N}}{\left(\int_M u^{\frac{2(m+n)}{m+n-2}} dV_{g_M\times g_N}\right)^{\frac{m+n-2}{m+n}}}.$$

We have some splitting

$$dV_{g_M\times g_N} = dV_{g_M} dV_{g_N}.$$

Note

$$R_{g_M\times g_N} = R_{g_M} + R_{g_N}$$
$$\left|\nabla_{g_M\times g_N} u\right|^2 = \left|\nabla_{g_M} u\right|^2 + \left|\nabla_{g_N} u\right|^2.$$

But constants $\frac{4(m+n-1)}{m+n-2}$, $\frac{2(m+n)}{m+n-2}$ make difficult.

(ii) Given a compact manifold M, we can consider

$$\{Y(M, g) : g \text{ is Riemman metric on } M\}.$$

But we don't know the characterization of this set. We know nothing about this.

1.3 Existence and uniqueness of the Yamabe problem when $Y(M, g) < 0$.

When $Y(M, g) < 0$, we want to consider the uniqueness of the Yamabe equation

$$-\frac{4(n-1)}{n-2} \triangle_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}},$$

where $\tilde{g} = u^{\frac{4}{n-2}} g$. Recall that if u is minimizer, then we have

$$R_{\tilde{g}} = \frac{E(u)}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{2}{n}}} = \frac{Y(M, g)}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{2}{n}}}.$$

So

$$-\frac{4(n-1)}{n-2} \triangle_g u + R_g u = \frac{Y(M, g)}{\left(\int_M u^{\frac{2n}{n-2}} dV_g\right)^{\frac{2}{n}}} u^{\frac{n+2}{n-2}}, \tag{1.9}$$

Then we have the following uniqueness result when $Y(M, g) < 0$.

Theorem 1.20. *If u, v satisfy* (1.9) *with $\int_M u^{\frac{2n}{n-2}} dV_g = \int_M v^{\frac{2n}{n-2}} dV_g$, then $u \equiv v$.*

Theorem 1.20 follows from the following theorem.

Theorem 1.21. *Let $g_1, g_2 \in [g_0]$ such that $R_{g_1} = R_{g_2} < 0$. Then $g_1 = g_2$.*

Proof. Since g_1, g_2 are in the same conformal class, g_1 is conformal to g_2. So $g_2 = u^{\frac{4}{n-2}} g_1$ for some $0 < u \in C^\infty(M)$. Now consider the Yamabe equation

$$-\frac{4(n-1)}{n-2} \triangle_{g_1} u + R_{g_1} u = R_{g_2} u^{\frac{n+2}{n-2}}.$$

Since $R_{g_1}=R_{g_2}$, we have

$$-\frac{4(n-1)}{n-2}\triangle_{g_1}u=R_{g_1}\left(u^{\frac{n+2}{n-2}}-u\right).$$

We claim that $\sup_M u\leq 1$. Since M is compact, there exists $x_0\in M$ such that $u(x_0)=\sup_M u$. Then

$$\triangle_{g_0}u(x_0)\leq 0.$$

At x_0, we have

$$-\frac{4(n-1)}{n-2}\triangle_{g_1}u(x_0)=R_{g_1}\left(u(x_0)^{\frac{n+2}{n-2}}-u(x_0)\right).$$

Since $R_{g_1}<0$, $u(x_0)^{\frac{n+2}{n-2}}-u(x_0)\leq 0$. Since $u>0$, $u(x_0)\leq 1$. Following the above proof, we have $\inf_M u\geq 1$. So $u=1$. This shows that $g_1=g_2$. □

Remark. (i) The condition in Theorem 1.21 cannot be weaken to $R_{g_1}=R_{g_2}\leq 0$ since the proof does not work anymore. In fact, we have the following counterexample: $M=\mathbb{T}^n$, $g_0=$flat metric on $\mathbb{T}^n$.

$$R_{g_0}=0=R_{cg_0}$$

for any positive constant c. But $g_0\neq cg_0$ if $c\neq 1$. The positive case has no uniqueness. Consider $\mathbb{S}^n$ with standard metric g_0. Then

$$R_{g_0}=\frac{n(n-1)}{2}.$$

Choose any isometry $\varphi\in O(n)$. Then $R_{g_0}=R_{\varphi^*g_0}$. But $g_0\neq\varphi^*g_0$ for $\varphi\neq\mathrm{id}$.

(ii) $R_{g_1}=R_{g_2}$ may not be constant. So we have also an uniqueness of prescribing curvature problem. The prescribing curvature problem is the following: Given a Riemannian manifold (M,g_0) and $f\in C^\infty(M)$, does there exists a $g\in[g_0]$ such that $R_g=f$?

Note that this is equivalent to

$$-\frac{4(n-1)}{n-2}\triangle_g u+R_g u=fu^{\frac{n+2}{n-2}}$$

when $n\geq 3$ and

$$-\triangle_g u+R_g=fe^{2u}$$

when $n=2$.

When $(M,g_0)=(\mathbb{S}^n,g_{\mathbb{S}^n})$, this is called *Nirenberg problem*. If f is constant, this reduces to the Yamabe problem.

By Theorem 1.21, if f is a negative function, then the metric g in the prescirbe curvature problem is unique. Of course, this is not solvable for some f.

Lemma 1.22. *Let $\mathbb{T}^n$ be a n-dimensional torus, g_0 be a flat metric. If $g\in[g_0]$ satisfying $R_g=f$, then f must change sign or $f\equiv 0$.*

Proof. First, consider $n\geq 3$. If $g=u^{\frac{4}{n-2}}g_0$ such that $R_g=f$, then

$$-\frac{4(n-1)}{n-2}\triangle_{g_0}u+R_{g_0}u=fu^{\frac{n+2}{n-2}}.$$

Integrate over $\mathbb{T}^n$. Then

$$0=\frac{4(n-1)}{n-2}\int_{\mathbb{T}^n}\triangle_{g_0}u\,dV_{g_0}=\int_{\mathbb{T}^n}fu^{\frac{n+2}{n-2}}dV_{g_0}.$$

For $n=2$, if $g=e^{2u}g_0$ with $R_g=f$, then

$$-\triangle_{g_0}u=fe^{2u}.$$

Integrate it. Then

$$0=-\int_{\mathbb{T}^2}\triangle_{g_0}udV_{g_0}=\int_{\mathbb{T}^2}fe^{2u}dV_{g_0}.$$

So we are done. □

Remark. When $n=2$, applying the Gauss-Bonnet theorem, we obtain the Lemma.

If $R_g=f$ and for $n=2$, $K_g=\frac{1}{2}R_g$. So $K_g=\frac{1}{2}f$. Hence by Gauss-Bonnet Theorem tells that

$$\frac{1}{2}\int_{\mathbb{T}^2}fdV_g=\int_{\mathbb{T}^2}K_gdV_g=2\pi\chi\left(\mathbb{T}^n\right)=2\pi\left(2-2\text{genus}\left(\mathbb{T}^n\right)\right)=0.$$

Theorem 1.23. *Let (M,g_0) be a compact smooth n-dimensional Riemannian manifold. Consider the semilineaer elliptic equation*

$$\triangle_{g_0}u+f\left(x,u\right)=0, \tag{1.10}$$

where $f\in C^\infty\left(M\times\mathbb{R}\right)$. If there exist $\varphi,\psi\in C^2\left(M\right)$ satisfying

$$\begin{aligned}\triangle_{g_0}\varphi+f\left(x,\varphi\right)&\geq 0 \qquad (1.11)\\ \triangle_{g_0}\psi+f\left(x,\psi\right)&\leq 0,\end{aligned}$$

i.e., φ and ψ are subsolution and supersolution of (1.10)*, respectively, such that $\varphi\leq\psi$, then* (1.10) *has a C^∞ solution u satisfying $\varphi\leq u\leq\psi$.*

Proof. Since M is compact, there exists a constant A such that $-A\leq\varphi\leq\psi\leq A$. Since M is compact, we can choose $c\gg 1$ such that for each $x\in M$, $F\left(x,t\right):=ct+f\left(x,t\right)$ so that F is increasing in t in $[-A,A]$. Define the operator

$$Lu=-\triangle_{g_0}u+cu.$$

Then $L:C^{2,\alpha}\left(M\right)\to C^{0,\alpha}\left(M\right)$ is an elliptic second order operator. By the Schauder theory, $L^{-1}:C^{0,\alpha}\left(M\right)\to C^{2,\alpha}\left(M\right)$ exists and it is a compact operator.

Claim 1. $Lv_1\geq Lv_2$, then $v_1\geq v_2$ (comparison principle)

Proof of Claim 1. By assumption, $L\left(v_1-v_2\right)\geq 0$. We claim that $\min_M\left(v_1-v_2\right)\geq 0$. Since M is compact and v_1,v_2 are continuous, there exists a point $x_0\in M$ such that

$$\left(v_1-v_2\right)\left(x_0\right)=\min_M\left(v_1-v_2\right).$$

Then for this x_0, we have $\triangle_{g_0}\left(v_1-v_2\right)\left(x_0\right)\geq 0$.

$$\begin{aligned}0&\leq L\left(v_1-v_2\right)\left(x_0\right)\\&=-\triangle_{g_0}\left(v_1-v_2\right)\left(x_0\right)+c\left(v_1-v_2\right)\left(x_0\right).\end{aligned}$$

So

$$c\left(v_1-v_2\right)\left(x_0\right)\geq\triangle_{g_0}\left(v_1-v_2\right)\left(x_0\right).$$

Since $c>0$, $v_1-v_2\left(x_0\right)\geq 0$, which proves the claim. □

Now we define φ_k and ψ_k inductively:

$$\begin{aligned} \varphi_k &= L^{-1}\left(F\left(x, \varphi_{k-1}\right)\right), \quad \text{for } k \geq 1 \quad \text{and} \quad \varphi_0 = \varphi \\ \psi_k &= L^{-1}\left(F\left(x, \psi_{k-1}\right)\right), \quad \text{for } k \geq 1 \quad \text{and} \quad \psi_0 = \psi \end{aligned} \tag{1.12}$$

Claim 2. $L\varphi_0 \leq L\varphi_1 \leq L\psi_1 \leq L\psi_0$.

By (1.12),

$$L\varphi_1 = F\left(x, \psi\right), \quad L\psi_1 = F\left(x, \psi\right).$$

By assumption, we know $\varphi \leq \psi$. Since F is increasing in t, $L\varphi_1 \leq L\psi_1$. Note that

$$\begin{aligned} L\varphi_1 - L\varphi_0 &= F\left(x, \varphi_0\right) - L\varphi_0 \\ &= \triangle_{g_0}\varphi + f\left(x, \varphi\right) \geq 0 \end{aligned}$$

by (1.11). Replacing φ with ψ in the above, we get $L\psi_1 - L\psi_0 \leq 0$. By Claim 1, we have

$$\varphi_0 \leq \varphi_1 \leq \psi_1 \leq \psi_0.$$

By induction, we have

$$\varphi_0 \leq \varphi_{k-1} \leq \varphi_k \leq \psi_k \leq \psi_{k-1} \leq \psi_0$$

for all k. Since $\{\varphi_k\}$ is a sequence of monotone increasing functions and bounded by ψ_0, there exists a pointwise limit of φ_k and write the limit by $\underline{u}$. Similarly, since $\{\psi_k\}$ is monotone decreasing and bounded below by φ_0, there exists $\overline{u}$ such that $\psi_k \to \overline{u}$ pointwise. Also, we have

$$u_0 \leq \underline{u} \leq \overline{u} \leq \psi_0.$$

By (1.12), $L\varphi_k = F\left(x, \varphi_{k-1}\right)$ for $k \geq 1$. Note that φ_k is bounded. Since F is smooth on M, $F(x, \varphi_{k-1}) \in L^p(M)$ for any $1 \leq p \leq \infty$. With this same p, $\varphi_k \in L^p\left(M\right)$. So $\varphi_k \in W^{1,p}(M)$. If we choose $p > n$, then by Sobolev embedding theorem, $\varphi_k \in C^{0,\alpha}(M)$ with

$$\|\varphi_k\|_{C^{0,\alpha}(M)} \leq C \|\varphi_k\|_{W^{1,p}(M)}.$$

Now from $|\varphi_k| \leq \varphi_k - \varphi_0 + |\varphi_0| \leq \psi_0 + 2|\varphi_0|$, we have

$$\|\varphi_k\|_{C^{0,\alpha}(M)} \leq C$$

where C is independent of k. Thus, by Arzelá-Ascoli's theorem, there exists a convergent subsequence, still denoted it by $\{\varphi_k\}$, such that $\varphi_k \to \tilde{u}$ in $C^{0,\alpha}(M)$. Then $\tilde{u} = \overline{u}$. So we obtain the pointwise convergence $\varphi_k \to \underline{u}$ is in fact $\varphi_k \to \underline{u}$ in $C^{0,\alpha}\left(M\right)$. Similarly, the pointwise convergence $\psi_k \to \overline{u}$ is in fact $\psi_k \to \overline{u}$ in $C^{0,\alpha}$.

By taking limit in $L\varphi_k = F\left(x, \varphi_{k-1}\right)$ and $L\psi_k = F\left(x, \psi_{k-1}\right)$, we have $L\left(\underline{u}\right) = F\left(x, \underline{u}\right)$ and $L\left(\overline{u}\right) = F\left(x, \overline{u}\right)$. Since $\underline{u}, \overline{u} \in C^{0,\alpha}(M)$, $F(x, \overline{u}), F(x, \overline{u}) \in C^{0,\alpha}(M)$. Now by the Schauder theory, $\underline{u}, \overline{u} \in C^{2,\alpha}\left(M\right)$. Now bootstrap argument yields $\underline{u}, \overline{u} \in C^{\infty}\left(M\right)$. By definition of L and F, we have

$$-\triangle_{g_0}\overline{u} = f\left(x, \overline{u}\right),$$

i.e., $\overline{u}$ is a solution for (1.10). This completes the proof. □

Based on this theorem, we prove the Yamabe problem for (M, g_0) with $Y\left(M, g_0\right) < 0$. The Yamabe equation takes this form

$$-\frac{4\left(n-1\right)}{n-2}\triangle_{g_0}u + R_{g_0}u = R_g u^{\frac{n+2}{n-2}},$$

where R_g is a negative constant. Since $Y(M,g)<0$, by Proposition 1.16, without loss of generality, we can assume $R_{g_0}<0$. Define

$$f(x,u)=\frac{n-2}{4(n-1)}\left[R_g u^{\frac{n+2}{n-2}}-R_{g_0}u\right].$$

Clearly, f is C^∞. To apply theorem 1.23, we want to find φ and ψ satisfying (1.11). Take $\varphi=\varepsilon$. Then

$$\begin{aligned} f(x,\varphi) &= \frac{n-2}{4(n-1)}\left(R_g\varepsilon^{\frac{n+2}{n-2}}-R_{g_0}\varepsilon\right) \\ &\geq \frac{n-2}{4(n-1)}\left(R_g\varepsilon^{\frac{n+2}{n-2}}-\max_M R_{g_0}\varepsilon\right)>0 \end{aligned}$$

if ε is sufficiently small. Take $\psi=C>0$. Then

$$\begin{aligned} f(x,\psi) &= \frac{n-2}{4(n-1)}\left(R_g C^{\frac{n+2}{n-2}}-R_{g_0}C\right) \\ &\leq \frac{n-2}{4(n-1)}\left(R_g C^{\frac{n+2}{n-2}}-\left(\min_M R_{g_0}\right)C\right)<0 \end{aligned}$$

if C is sufficiently large. So at the same time, $\varphi\leq\psi$. Now apply Theorem 1.23, we get a C^∞-solution u to the Yamabe equation with $u>0$.

Remark. Try it for (M,g_0) with $Y(M,g_0)>0$. Then we cannot use Theorem 1.23.

2 Yamabe problem on non-locally conformally flat manifold

2.1 Aubin-Trudinger approach to the Yamabe problem

So far we consider the Yamabe problem when $Y(M, g_0) < 0$. From now on, we focus on the case $Y(M, g_0) \geq 0$.

Theorem 2.1 (Aubin-Trudinger)**.** *For a compact n-dimensional Riemannian manifold (M, g_0),*

(i) *$Y(M, g_0) \leq Y(\mathbb{S}^n, g_{\mathbb{S}^n})$, and*

(ii) *if $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$, then the Yamabe problem is solvable.*

We want to compute $Y(\mathbb{S}^n, g_{\mathbb{S}^n})$. In order to do this, we would like to talk about the best constant of Sobolev inequality of $\mathbb{R}^n$. The Sobolev inequality of $\mathbb{R}^n$ tells us that there exists a constant Λ depending only on n such that

$$\Lambda \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \tag{2.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Here $C_0^\infty(\mathbb{R}^n)$ is the set of all smooth functions in $\mathbb{R}^n$ with compact support.

Question. What is the smallest number Λ such that (2.1) is true?

This question is closely related to Yamabe problem. Note that the scalar curvature of $\mathbb{R}^n$ with respect to the standard metric is 0. Therefore, the conformal Laplacian of $\mathbb{R}^n$ is just the standard Laplacian up to a constant:

$$L_0 = -\frac{4(n-1)}{n-2} \triangle.$$

The best constant in (2.1) is

$$\begin{aligned} \Lambda &= \inf \left\{ \frac{\frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}} \,\middle|\, u \in C_0^\infty(\mathbb{R}^n), u \not\equiv 0 \right\} \\ &= \inf \left\{ E(u) : u \in C_0^\infty(\mathbb{R}^n), u \not\equiv 0 \right\} \\ &= Y(\mathbb{R}^n, g_{\text{flat}}), \end{aligned}$$

which is the Yamabe constant of $(\mathbb{R}^n, g_{\text{flat}})$.

Let Ω be a bounded open set in $\mathbb{R}^n$. We can define

$$\Lambda(\Omega) = \inf \left\{ E(u) : 0 \not\equiv u \in C_0^\infty(\Omega) \right\}.$$

Lemma 2.2. $\Lambda(\Omega) = \Lambda$.

Proof. Consider $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ $\varphi(x) = x + a$ for some fixed $a \in \mathbb{R}^n$, which is the translation in $\mathbb{R}^n$. If Ω and Ω' are open sets in $\mathbb{R}^n$ such that $\Omega' = \varphi(\Omega)$, $\Lambda(\Omega) = \Lambda(\Omega')$ because φ is an isometry of g_{flat}.

If Ω and Ω' are open sets in $\mathbb{R}^n$ satisfying $\Omega \subset \Omega'$, then $\Lambda(\Omega) \geq \Lambda(\Omega')$ since $C_0^\infty(\Omega) \subset C_0^\infty(\Omega')$. Given any bounded open set Ω in $\mathbb{R}^n$, there exists a translation φ such that $0 \in \varphi(\Omega)$. Since $\varphi(\Omega)$ is open and bounded,

$$B_{r_1}(0) \subset \varphi(\Omega) \subset B_{r_2}(0)$$

for some $r_1, r_2 > 0$. So

$$\Lambda\left(B_{r_2}(0)\right) \leq \Lambda(\Omega) \leq \Lambda\left(B_{r_1}(0)\right).$$

Claim 1. For any $r > 0$, $\Lambda\left(B_r(0)\right) = \Lambda_0$.

Claim 2. $\Lambda_0 = \Lambda = Y\left(\mathbb{R}^n, g_{\mathbb{R}^n}\right)$.

Proof of Claim 1. It suffices to prove $\Lambda\left(B_r\right) = \Lambda\left(B_1\right)$ for all $r > 0$. To see this, let $0 \not\equiv u \in C_0^\infty\left(B_r(0)\right)$. Now we define

$$\tilde{u}(x) := u(rx) \quad \text{for } x \in B_1(0).$$

Then $0 \not\equiv \tilde{u} \in C_0^\infty\left(B_1(0)\right)$. Then $\nabla\tilde{u}(x) = r(\nabla u)(rx)$. So change of variable yields

$$\begin{aligned} E(\tilde{u}) &= \frac{\frac{4(n-1)}{n-2}\int_{B_1(0)} |\nabla\tilde{u}|^2\, dx}{\left(\int_{B_1(0)} \tilde{u}^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}} \\ &= E(u), \end{aligned}$$

which proves Claim 1. □

Proof of Claim 2. Since $C_0^\infty\left(B_r(0)\right) \subset C_0^\infty\left(\mathbb{R}^n\right)$, $\Lambda_0 \geq \Lambda$. Note that

$$\bigcup_{r>0} B_r(0) = \mathbb{R}^n.$$

So $\bigcup_{r>0} C_0^\infty\left(B_r\right)$ is dense in $C_0^\infty\left(\mathbb{R}^n\right)$.

For all i, there exists $u_i \in C_0^\infty\left(\mathbb{R}^n\right)$ such that $E\left(u_i\right) \leq \Lambda + \frac{1}{2i}$. So there exists $\tilde{u}_i \in C_0^\infty\left(B_{r_i}\right)$ such that

$$E\left(\tilde{u}_i\right) \leq E\left(u_i\right) + \frac{1}{2i} \leq \Lambda + \frac{1}{i}.$$

So

$$\Lambda_0 = \Lambda\left(B_{r_i}(0)\right) \leq E(\tilde{u}) \leq \Lambda + \frac{1}{i}.$$

Now letting $i \to \infty$, we are done. □

Hence by Claim 1 and Claim 2, we completes the proof of Lemma. □

Consider

$$\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$

We have the stereographic projection from the nouth pole $N = (0, \ldots, 1) \in \mathbb{S}^n$

$$\begin{aligned} &\pi : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \\ &\pi\left(x_1, \ldots, x_n\right) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right), \end{aligned}$$

where $x = \left(x_1, \ldots, x_n, x_{n+1}\right) \in \mathbb{S}^n \setminus \{N\}$.

π is bijective, diffeomorphism,

$$\begin{aligned} &\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}, \\ &\pi^{-1}\left(\varphi_1, \ldots, \varphi_n\right) = \left(\frac{2\varphi_1}{|\varphi|^2 + 1}, \ldots, \frac{2\varphi_n}{|\varphi|^2 + 1}, \frac{|\varphi|^2 - 1}{|\varphi|^2 + 1}\right), \end{aligned}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{R}^n$.

The standard Riemmanian metric $g_{\mathbb{S}^n}$ on $\mathbb{S}^n$ is given by $g_{\mathbb{S}^n} = i^* (g_{\mathbb{R}^n})$, where $i : \mathbb{S}^n \to \mathbb{S}^{n+1}$, $i(x) = x$.

In particular, $\pi^{-1} : (\mathbb{R}^n, g_{\mathbb{R}^n}) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$. In particular, π^{-1} is a diffeomorphism. We can compute $\left(\pi^{-1}\right)^* g_{\mathbb{S}^n}$. We will see π^{-1} is a conformal map in the sense that $\left(\pi^{-1}\right)^* g_{\mathbb{S}^n} = f g_{\mathbb{R}^n}$.

We want to show that

$$\left(\pi^{-1}\right)^* (g_{\mathbb{S}^n}) = \left(\frac{2}{1+|\varphi|^2}\right)^2 g_{\mathbb{R}^n}.$$

At $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{R}^n$, $\left(\frac{\partial}{\partial \varphi_1}, \ldots, \frac{\partial}{\partial \varphi_n}\right)$ is a basis of $T_\varphi \mathbb{R}^n$. Consider

$$\begin{aligned}
\left(\pi^{-1}\right)^* (g_{\mathbb{S}^n}) \left(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_j}\right) &= g_{\mathbb{S}^n}\left(d\left(\pi^{-1}\right)\left(\frac{\partial}{\partial \varphi_i}\right), d\left(\pi^{-1}\right)\left(\frac{\partial}{\partial \varphi_i}\right)\right) \\
&= g_{\mathbb{S}^n}\left(\frac{\partial \pi^{-1}}{\partial \varphi_i}, \frac{\partial \pi^{-1}}{\partial \varphi_j}\right) \\
&= i^* g_{\mathbb{R}^{n+1}}\left(\frac{\partial \pi^{-1}}{\partial \varphi_i}, \frac{\partial \pi^{-1}}{\partial \varphi_j}\right) \\
&= g_{\mathbb{R}^{n+1}}\left(\frac{\partial \pi^{-1}}{\partial \varphi_i}, \frac{\partial \pi^{-1}}{\partial \varphi_j}\right)
\end{aligned}$$

Note

$$\frac{\partial \pi^{-1}}{\partial \varphi_i} = \left(-\frac{4\varphi_1\varphi_i}{\left(|\varphi|^2+1\right)^2}, \ldots, \frac{2}{|\varphi|^2+1} - \frac{4\varphi_i^2}{\left(|\varphi|^2+1\right)^2}, \ldots, -\frac{4\varphi_n\varphi_i}{\left(|\varphi|^2+1\right)^2}, \frac{4\varphi_i}{\left(|\varphi|^2+1\right)^2}\right).$$

If $i = j$, then

$$\begin{aligned}
\left(\pi^{-1}\right)^* (g_{\mathbb{S}^n}) \left(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_j}\right) &= \left(\frac{-4\varphi_1\varphi_i}{\left(1+|\varphi|^2\right)^2}\right)^2 \\
&\quad + \cdots + \left(\frac{2}{|\varphi|^2+1} - \frac{4\varphi_i^2}{\left(|\varphi|^2+1\right)^2}\right)^2 \\
&\quad + \cdots + \left(-\frac{4\varphi_n\varphi_i}{\left(|\varphi|^2+1\right)^2}\right)^2 + \left(\frac{4\varphi_i}{\left(|\varphi|^2+1\right)^2}\right)^2 \\
&= \sum_{j=1}^{n} \frac{16\varphi_j^2\varphi_i^2}{\left(1+|\varphi|^2\right)^4} + \frac{4}{\left(|\varphi|^2+1\right)^2} - \frac{16\varphi_i^2}{\left(1+|\varphi|^2\right)^3} + \frac{16\varphi_i^2}{\left(1+|\varphi|^2\right)^4} \\
&= \frac{16\left(|\varphi|^2+1\right)\varphi_i^2}{\left(1+|\varphi|^2\right)^4} + \frac{4}{\left(|\varphi|^2+1\right)^2} - \frac{16\varphi_i^2}{\left(|\varphi|^2+1\right)^3} = \frac{4}{\left(|\varphi|^2+1\right)^2}.
\end{aligned}$$

If $i \neq j$, then

$$\left(\pi^{-1}\right)^* (g_{\mathbb{S}^n}) \left(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_j}\right) = \frac{16\varphi_1^2\varphi_i\varphi_j}{\left(1+|\varphi|^2\right)^4} + \cdots + \left(\frac{2}{|\varphi|^2+1} - \frac{4\varphi_i^2}{\left(|\varphi|^2+1\right)^2}\right)\left(-\frac{4\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^2}\right)$$

$$+\cdots+\left(\frac{2}{|\varphi|^2+1}-\frac{4\varphi_i^2}{\left(|\varphi|^2+1\right)^2}\right)\left(\frac{-4\varphi_j\varphi_i}{\left(|\varphi|^2+1\right)^2}\right)$$
$$+\cdots+\frac{16\varphi_n^2\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^4}+\frac{16\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^4}$$
$$=\frac{16\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^4}\sum_{k=1}^n\varphi_k^2-\frac{16\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^3}+\frac{16\varphi_i\varphi_j}{\left(|\varphi|^2+1\right)^4}=0.$$

We have shown that

$$\left(\pi^{-1}\right)^* g_{\mathbb{S}^n}=\frac{4}{\left(|\varphi|^2+1\right)^2}g_{\mathbb{R}^n}. \tag{2.2}$$

That is to say, $\pi^{-1}:\mathbb{R}^n\to\mathbb{S}^n\setminus\{N\}$ is a conformal map.

Definition 2.3. Given a map $f:(M,g_M)\to(N,g_N)$, if f is a diffeomorphism and f^*g_N is conformal to g_M, then f is said to be a *conformal diffeomorphism*.

Proposition 2.4. *If $f:(M,g_M)\to(N,g_N)$ is a conformal diffeomorphism, then $Y(M,g_M)=Y(N,g_N)$.*

Proof. Since M and N are diffeomorphic, $\dim M=\dim N=n$. Let $0<u\in C_0^\infty(N)$. Then change of variable gives

$$E_{g_N}(u)=\frac{\int_N\frac{4(n-1)}{n-2}|\nabla_{g_N}u|^2+R_{g_N}u\,dV_{g_N}}{\left(\int_N|u|^{\frac{2n}{n-2}}dV_{g_N}\right)^{\frac{n-2}{n}}}$$
$$=\frac{\int_M\frac{4(n-1)}{n-2}|\nabla_{f^*g_N}(u\circ f)|^2+R_{f^*g_N}(u\circ f)^2\,dV_{f^*g_N}}{\left(\int_M(u\circ f)^{\frac{2n}{n-2}}dV_{f^*g_N}\right)^{\frac{n-2}{n}}}.$$

Since f is a conformal diffeomorphism, $f^*g_N\in[g_M]$. So

$$\inf_{0<u\in C_0^\infty(N)}E(u)=\inf_{0<u\in C_0^\infty(N)}E_{g_M}(u\circ f)=\inf_{0<u\in C_0^\infty(M)}E(u).$$

This completes the proof of Proposition. □

Applying Proposition 2.4, we have

$$\begin{aligned}Y(\mathbb{R}^n,g_{\mathbb{R}^n})&=Y(\mathbb{S}^n\setminus\{N\},g_{\mathbb{S}^n})\\&=\inf\{E(u):0\not\equiv u\in H^1(\mathbb{S}^n\setminus\{N\})\}\\&=\inf\{E(u):0\not\equiv u\in H^1(\mathbb{S}^n)\}\\&=Y(\mathbb{S}^n,g_{\mathbb{S}^n}).\end{aligned}\tag{2.3}$$

Recall that $Y(\mathbb{R}^n,g_{\mathbb{R}^n})=\Lambda$, where Λ is the best constant in the Sobolev inequality.

Exercise 2.5. If $\tilde g=ug$ for some $0<u\in C^\infty(M)$, then prove that the Ricci curvature of $\tilde g$ and g are related by

$$\widetilde{\mathrm{Ric}}(X_i,X_k)$$

$$= \frac{n-2}{4}(\log u)_k(\log u)_i - \frac{n-2}{2}(\log u)_{ki} + \mathrm{Ric}(X_i, X_k) - \left(\frac{n-2}{4}|\nabla_g \log u|_g^2 + \frac{1}{2}\triangle_g(\log u)\right) g_{ik}.$$

Theorem 2.6 (Obata)**.** *If g is a metric on $\mathbb{S}^n$ conformal to $g_{\mathbb{S}^n}$ and has constant scalar curvature, then up to a constant scalar factor, g is obtained from $g_{\mathbb{S}^n}$ by a conformal diffeomorphism of $\mathbb{S}^n$.*

If $g \in [g_{\mathbb{S}^n}]$ with $R_g \equiv$ constant, then $g = cf^* g_{\mathbb{S}^n}$ for some constant $c > 0$ and $f \in \mathrm{Conf}(\mathbb{S}^n)$.

Proof. We show that g is Einstein, i.e., $\mathrm{Ric}_g - \frac{R_g}{n} g \equiv 0$, i.e., $B_{ij} = \mathrm{Ric}_{ij} - \frac{R_g}{n} g_{ij} = 0$. Here B_{ij} denotes the trace-less Ricci tensor. Write $g_{\mathbb{S}^n} = ug$. Then

$$\begin{aligned}(B_{g_{\mathbb{S}^n}})_{ij} &= (\mathrm{Ric}_{g_{\mathbb{S}^n}})_{ij} - \frac{R_{g_{\mathbb{S}^n}}}{n}(g_{\mathbb{S}^n})_{ij} \\ &= \frac{n-2}{4}(\log u)_i(\log u)_j - \frac{n-2}{2}(\log u)_{ij} + \mathrm{Ric}_{ij} - \left(\frac{n-2}{4}|\nabla_g \log u|_g^2 + \frac{1}{2}\triangle_g(\log u)\right) g_{ij} \\ &\quad - \frac{1}{n}u^{-1}\left[-\frac{1}{4}(n-2)(n-1)|\nabla_g \log u|_g^2 - (n-1)\triangle_g(\log u) + R_g\right] u g_{ij} \\ &= B_{ij} + \frac{n-2}{4}(\log u)_i(\log u)_j - \frac{n-2}{2}(\log u)_{ij} + \frac{n-2}{4}\left(\frac{n-1}{n} - 1\right)|\nabla_g \log u|_g^2 g_{ij} \\ &\quad + \left(\frac{n-1}{n} - \frac{1}{2}\right)\triangle_g(\log u) g_{ij}.\end{aligned}$$

Let $u = \varphi^{-2}$. Then $\log u = -2\log\varphi$ and

$$(\log u)_i = -\frac{2\varphi_i}{\varphi}, \quad (\log u)_{ij} = \left(-\frac{2\varphi_i}{\varphi}\right)_j = \frac{-2\varphi_{ij}}{\varphi} + \frac{2\varphi_i\varphi_j}{\varphi^2}.$$

So

$$|\nabla \log u|^2 = \frac{4|\nabla\varphi|^2}{\varphi^2}, \quad \triangle(\log u) = -\frac{2\triangle\varphi}{\varphi} + \frac{2|\nabla\varphi|^2}{\varphi^2}.$$

Thus,

$$(B_{g_{\mathbb{S}^n}})_{ij} = B_{ij} + \frac{n-2}{\varphi}\left(\varphi_{ij} - \frac{1}{n}\triangle\varphi g_{ij}\right). \tag{2.4}$$

Note $(B_{g_{\mathbb{S}^n}})_{ij} = 0$. Since R_g is constant, by the contracted Bianchi identity, we have

$$0 = \langle X, \nabla R_g\rangle = 2\sum_{i=1}^n X_i \,\mathrm{Ric}_g(X, X_i) \quad \text{for all vector field } X.$$

In particular,

$$0 = \sum_{i=1}^n X_i \,\mathrm{Ric}_g(X_j, X_i) = \sum_{i=1}^n X_i \,\mathrm{Ric}_{ji} \quad \text{for all } j. \tag{2.5}$$

So

$$\begin{aligned}\sum_{i=1}^n X_i B_{ji} &= \sum_{i=1}^n X_i\left(\mathrm{Ric}_{ji} - \frac{R_g}{n} g_{ji}\right) \\ &= 0\end{aligned}$$

since R_g is constant and X_i is compatible with the metric g_{ij}. Therefore, by we have

$$\begin{aligned}\int_{\mathbb{S}^n} |B|_g^2 \varphi dV_g &= \int_{\mathbb{S}^n} \varphi B^{ij} B_{ij} dV_g \\ &= -(n-2)\int_{\mathbb{S}^n} \left(B^{ij}\right)\left(\varphi_{ij} - \frac{1}{n}\triangle\varphi g_{ij}\right) dV_g && \text{by (2.4)} \\ &= -(n-2)\int_{\mathbb{S}^n} B^{ij}\varphi_{ij} dV_g && \text{since } B_{ij} \text{ is traceless} \\ &= -(n-2)\int_{\mathbb{S}^n} \left(B^{ij}\right)_j \varphi_i dV_g && \text{integration by parts} \\ &= 0 && \text{by (2.5).}\end{aligned}$$

Since φ is arbitrary, $B = 0$, i.e., g is Einstein.

We need two facts. Let us assume these facts in the moment.

Fact 1. Since g is conformal to $g_{\mathbb{S}^n}$ and $g_{\mathbb{S}^n}$ is *locally conformally flat,* g is also locally conformally flat. So the Weyl tensor of g, $W \equiv 0$.

Fact 2. If $B \equiv 0$ and $W \equiv 0$, then g has constant sectional curvature.

By Fact 1 and Fact 2, $(\mathbb{S}^n, g)$ is isometric up to constant factor to $(\mathbb{S}^n, g_{\mathbb{S}^n})$, i.e., there exists an isometry $f : (\mathbb{S}^n, g) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ such that $g = cf^* g_{\mathbb{S}^n}$. Since $g = \varphi^{-2} g_{\mathbb{S}^n} = cf^* g_{\mathbb{S}^n}$, we are done. ☐

First, we explain Fact 1. We define the following terminology which we used in Fact 1.

Definition 2.7. We say (M^n, g) is *locally conformally flat* if it is locally conformal to the flat metric of $\mathbb{R}^n$, i.e., for any $x \in M^n$, there is a neighborhood U of x and a parametrization $f : V \subset \mathbb{R}^n \to U$ such that $f^* g = u g_{\mathbb{R}^n}$ for some $u > 0$.

Example 2.8. For $0 < u \in C^\infty(\mathbb{R}^n)$, $(\mathbb{R}^n, u g_{\mathbb{R}^n})$ is locally conformally flat (in fact, it is global conformally flat, or conformally flat.)

Example 2.9. $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is locally conformally flat. Indeed, for any $x \in \mathbb{S}^n \setminus \{N\}$, $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$, inverse of stereographic projection. Then

$$\left(\pi^{-1}\right)^* g_{\mathbb{S}^n} = \frac{4}{\left(1 + |\varphi|^2\right)^2} g_{\mathbb{R}^n}.$$

Definition 2.10. Given a Riemannian metric g, the *Weyl tensor* of g is given by

$$\begin{aligned}W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(\mathrm{Ric}_{ik}\, g_{jl} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{il}\, g_{jk} - \mathrm{Ric}_{jk}\, g_{il}\right) \\ &+ \frac{1}{(n-1)(n-2)} R_g \left(g_{ik}g_{jl} - g_{il}g_{jk}\right).\end{aligned}$$

If h, m are $(0,2)$-tensors, we define

$$(h \owedge m)_{ijkl} = h_{ik}m_{jl} + h_{jl}m_{ik} - h_{il}m_{jk} - h_{jk}m_{il}.$$

We call $(h \owedge m)$ the *Kulkarni-Nomizu product*.

$$W = Rm - \frac{1}{n-2}\mathrm{Ric} \owedge g + \frac{R_g}{2(n-1)(n-2)} g \owedge g.$$

The following proposition explains some properties of Weyl tensors.

Proposition 2.11.

(i) *Weyl tensor satisfies all the algebraic identities of Riemann curvature tensor*

(a) $W_{ijkl} = -W_{jikl}$.

(b) $W_{ijkl} = -W_{ijlk}$.

(c) $W_{ijkl} = W_{klij}$.

(d) $W_{ijkl} + W_{jkil} + W_{kijl} = 0$.

(ii) *Weyl tensor is a pointwise conformal invariant. More precisely, if $\tilde{g} = e^{2u} g$, then*

$$\tilde{W}_{ijkl} = e^{-2u} W_{ijkl}.$$

(iii) *g is locally conformally flat if and only if the Weyl tensor of g is equal to* 0.

Proof. (i) is easy.
(ii) We know $\widetilde{Rm}$ and Rm, $\widetilde{\mathrm{Ric}}$ and Ric, and $\tilde{R}_g$ and $\tilde{R}_g$ change conformally, just put it these.
(iii) ($\Rightarrow$) Easy by computation ($\Leftarrow$): difficult. □

As a consequence of Proposition (2.11), if $\tilde{g} \in [g]$ and g is locally conformally flat, then $W_g = 0$ by (iii) and so $W_{\tilde{g}} = e^{-2u} W_g \equiv 0$ by (ii). Thus, $\tilde{g}$ is locally conformally flat by (iii).
We prove Fact 2.

Proof. If $B \equiv 0$, then by definition

$$\mathrm{Ric}_{ij} = \frac{R_g}{n} g_{ij}.$$

From this and the contractive Binachi's identity, we have

$$\langle X, \nabla R_g \rangle = 2 \sum_{i=1}^{n} X_i \,\mathrm{Ric}_g(X, X_i) = 2 \sum_{i=1}^{n} X_i \left(\frac{R_g}{n} \right) g(X, X_i).$$

Put $X = X_j$. Then we have

$$X_j(R_g) = \langle X_j, \nabla R_g \rangle = \frac{2}{n} X_j(R_g) \quad \text{for all } j.$$

Since $n \geq 3$, this shows that R_g is constant.
Now

$$\begin{aligned}
0 &\equiv W_{ijkl} \\
&= R_{ijkl} - \frac{1}{n-2} \left(\mathrm{Ric}_{ik}\, g_{jl} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{il}\, g_{jk} - \mathrm{Ric}_{jk}\, g_{il} \right) \\
&\quad + \frac{1}{(n-1)(n-2)} R_g \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \\
&= R_{ijkl} + \left(-\frac{2}{n(n-2)} + \frac{1}{(n-1)(n-2)} \right) R_g \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \\
&= R_{ijkl} - \frac{R_g}{n(n-1)} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right).
\end{aligned}$$

Thus, by Schur's Lemma, (M, g) has constant sectional curvature. It is equal to $\frac{R_g}{n(n-1)}$. □

Exercise 2.12. If g_1, g_2 are locally conformally flat, then is $g_1 \times g_2$ is locally conformal flat? If g is locally conformally flat, Ricci soliton, prove that g has constant sectional curvature.

Exercise 2.13. Find a Riemannian metric g which is not locally conformally flat.

Now using Obata's theorem(Theorem 2.6), we can now compute $Y(\mathbb{S}^n, g_{\mathbb{S}^n})$.

Proposition 2.14.

(i) $Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ *is attained. That is, there exists* $0 < u \in C^\infty(\mathbb{S}^n)$ *such that* $E(u) = Y(\mathbb{S}^n, g_{\mathbb{S}^n})$.

(ii) $Y(\mathbb{S}^n, g_{\mathbb{S}^n}) = n(n-1)\operatorname{Vol}(\mathbb{S}^n, g_{\mathbb{S}^n})^{\frac{2}{n}}$.

First we prove (ii) first assuming (i). By (i), there exists $0 < u \in C^\infty(M)$ such that

$$E(u) = Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

By Proposition 1.8, $g = u^{\frac{4}{n-2}} g_{\mathbb{S}^n}$ has constant scalar curvature. Hence by Obata Theorem, g has constant sectional curvature and $g = cf^* g_{\mathbb{S}^n}$. By (1.6),

$$\begin{aligned} E(u) &= \frac{\int_{\mathbb{S}^n} R_g dV_g}{\left(\int_{\mathbb{S}^n} dV_g\right)^{\frac{n-2}{n}}} \\ &= \frac{\int_{\mathbb{S}^n} R_{cf^* g_{\mathbb{S}^n}} dV_{g_{\mathbb{S}^n}}}{\left(\int_{\mathbb{S}^n} dV_{cf^* g_{\mathbb{S}^n}}\right)^{\frac{n-2}{n}}} && \text{since } g = cf^* g_{\mathbb{S}^n} \\ &= \frac{\int_{\mathbb{S}^n} R_{g_{\mathbb{S}^n}} dV_{g_{\mathbb{S}^n}}}{\left(\int_{\mathbb{S}^n} dV_{g_{\mathbb{S}^n}}\right)^{\frac{n-2}{n}}} && \text{since } f \text{ is conformal diffeomorphism} \\ &= n(n-1)\left(\int_{\mathbb{S}^n} dV_{g_{\mathbb{S}^n}}\right)^{\frac{2}{n}}. \end{aligned}$$

This completes the proof of (ii).

Combining Lemma 2.2 and (2.3), we have

$$n(n-1)\operatorname{Vol}(\mathbb{S}^n, g_{\mathbb{S}^n}) = Y(\mathbb{S}^n, g_{\mathbb{S}^n}) = Y(\mathbb{R}^n, g_{\mathbb{R}^n}) = \Lambda = \Lambda(\Omega),$$

where Λ is the best constant in the Sobolev inequality.

Recall the Sobolev inequality in $\mathbb{R}^n$ tells us: there exists a constant Λ depending only on n such that

$$\Lambda\left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \le \frac{4(n-1)}{n-2}\int_{\mathbb{R}^n} |\nabla u|^2 dx \tag{2.6}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. We know that the smallest constant such that (2.6) holds is equal to $n(n-1)\operatorname{Vol}(\mathbb{S}^n, g_{\mathbb{S}^n})$.

Now we prove (i). We want to find a function u such that the equality holds in (2.6). Given $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$, we consider the function $u_{(\xi,\varepsilon)} : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u_{(\xi,\varepsilon)}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}}.$$

Take $\xi = 0 \in \mathbb{R}^n$, then $u_{(0,\varepsilon)}(x)$ is rotationally symmetric. **[DRAW A GRAPH]** The function $u_{(\xi,\varepsilon)}$ concentrates near ξ. The smaller ε is, the more concentrated at ξ $u_{(\xi,\varepsilon)}$ is.

Note

$$\frac{\partial u_{(\xi,\varepsilon)}}{\partial x_i} = \frac{\varepsilon^{\frac{n-2}{2}}}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}}}\left(-\frac{n-2}{2}\right)2\left(x_i-\xi_i\right) = \frac{-\left(n-2\right)\varepsilon^{\frac{n-2}{2}}\left(x_i-\xi_i\right)}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}}}$$

and

$$\frac{\partial^2 u_{(\xi,x)}}{\partial x_i \partial x_j} = -\frac{\left(n-2\right)\varepsilon^{\frac{n-2}{2}}}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}}}\delta_{ij} - \frac{\left(n-2\right)\varepsilon^{\frac{n-2}{2}}\left(x_i-\xi_i\right)\left(-\frac{n}{2}\right)2\left(x_j-\xi_j\right)}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}+1}} = -\frac{\left(n-2\right)\varepsilon^{\frac{n-2}{2}}}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}}}\delta_{ij} + \frac{n\left(n-2\right)\varepsilon^{\frac{n-2}{2}}\left(x_i-\xi_i\right)\left(x_j-\xi_j\right)}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}+1}}.$$

So

$$\triangle u_{(\xi,\varepsilon)} = -\frac{n\left(n-2\right)\varepsilon^{\frac{n-2}{2}}}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}}} + \frac{n\left(n-2\right)\varepsilon^{\frac{n-2}{2}}|x-\xi|^2}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}+1}} = -\frac{n\left(n-2\right)\varepsilon^{\frac{n-2}{2}}\varepsilon^2}{\left(\varepsilon^2+|x-\xi|^2\right)^{\frac{n}{2}+1}} = -n\left(n-2\right)u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}},$$

i.e., $\triangle u_{(\xi,\varepsilon)} + n\left(n-2\right)u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} = 0$. Multiply $u_{(\xi,\varepsilon)}$ to this equation and integrate, we get

$$\int_{\mathbb{R}^n}\left|\nabla u_{(\xi,\varepsilon)}\right|^2 dx = n\left(n-2\right)\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}dx,$$

i.e.,

$$\frac{4\left(n-1\right)}{n-2}\int_{\mathbb{R}^n}\left|\nabla u_{(\xi,\varepsilon)}\right|^2 dx = 4n\left(n-1\right)\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}dx.$$

So

$$E\left(u_{(\xi,\varepsilon)}\right) = \frac{\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n}\left|\nabla u_{(\xi,\varepsilon)}\right|^2 dx}{\left(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}} = n\left(n-1\right)\left(2^n\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}dx\right)^{\frac{2}{n}}.$$

Note

$$\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}dx = \int_{\mathbb{R}^n}\left(\frac{\varepsilon}{\varepsilon^2+|x-\xi|^2}\right)^{\frac{n-2}{2}\frac{2n}{n-2}}dx$$

$$= \int_{\mathbb{R}^n} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^n dx$$
$$= \int_{\mathbb{R}^n} \frac{1}{\left(1 + |y|^2\right)^n} dy.$$

So

$$E\left(u_{(\xi,\varepsilon)}\right) = n\,(n-1) \left(\int_{\mathbb{R}^n} \left(\frac{2}{1+|y|^2} \right)^n dy \right)^{\frac{2}{n}}.$$

On the other hand, we have

$$\begin{aligned}
\mathrm{Vol}\,(\mathbb{S}^n, g_{\mathbb{S}^n}) &= \int_{\mathbb{S}^n} dV_{g_{\mathbb{S}^n}} = \int_{\mathbb{S}^n \setminus \{N\}} dV_{g_{\mathbb{S}^n}} \\
&= \int_{\pi^{-1}(\mathbb{R}^n)} dV_{g_{\mathbb{S}^n}} \\
&= \int_{\mathbb{R}^n} \left(\pi^{-1}\right)^* dV_{g_{\mathbb{S}^n}} \\
&= \int_{\mathbb{R}^n} dV_{(\pi^{-1})^* g_{\mathbb{S}^n}} \\
&= \int_{\mathbb{R}^n} dV_{\frac{4}{(1+|y|^2)^2} g_{\mathbb{R}^n}} \\
&= \left(\int_{\mathbb{R}^n} \left(\frac{2}{1+|y|^2} \right)^n dy \right)^{\frac{2}{n}}.
\end{aligned}$$

This completes the proof.

Combining all these, we have

$$\begin{aligned}
E\left(u_{(\xi,\varepsilon)}\right) &= n\,(n-1) \left(\mathrm{Vol}\,(\mathbb{S}^n, g_{\mathbb{S}^n})\right)^{\frac{2}{n}} \\
&= Y\left(\mathbb{S}^n, g_{\mathbb{S}^n}\right) \\
&= Y\left(\mathbb{R}^n, g_{\mathbb{R}^n}\right) \\
&= \Lambda.
\end{aligned}$$

Remark. (i) $u_{(\xi,\varepsilon)}$ is not compactly supported. However, since $u_{(\xi,\varepsilon)}$concentrated at ξ and it has a fast decay, we can multiply a cut off function to $u_{(\xi,\varepsilon)}$ and the resulting function is compactly supported and its energy is arbitrary closed to Λ.

(ii) In fact, if u is a function in $\mathbb{R}^n$ such that $E(u) = \Lambda$, then $u = u_{(\xi,\varepsilon)}$ for some $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$.

All of these calculations have meaning. Recall the Aubin-Trudinger theorem. Given (M, g), the theorem tells that (i)$Y(M, g_0) \leq Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ and (ii) if $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$, then the Yamabe problem is solvable. So we need to know the exact value of $Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ since we have a 'strict inequality'.

Now we are ready to prove the Aubin-Trudinger theorem.

Proof of Theorem 2.1 (i). For any $\alpha > 0$, let B_α be the ball of radius α in $\mathbb{R}^n$ centered at the origin. Let

$$u_{(0,\xi)}(x) = \left(\frac{\varepsilon}{\varepsilon + |x|^2} \right)^{\frac{n-2}{2}}, \quad \varepsilon > 0.$$

Now choose a smooth cut-off radial function $0 \leq \eta \leq 1$ supported in $B_{2\alpha}$with $\eta \equiv 1$ on B_α, $|\nabla\eta| \leq \frac{2}{\alpha}$. Now consider $\varphi = \eta u_{(0,\xi)}$. Then

$$\begin{aligned}
\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n} |\nabla\varphi|^2\, dx &= \frac{4(n-1)}{n-2}\int_{B_{2\alpha}} |\nabla\varphi|^2\, dx \\
&= \frac{4(n-1)}{n-2}\int_{B_{2\alpha}} \eta^2 \left|\nabla u_{(0,\xi)}\right|^2 dx \\
&\quad + \frac{4(n-1)}{n-2}\int_{B_{2\alpha}} 2\eta u_{(0,\varepsilon)} \left\langle \nabla\eta, \nabla u_{(0,\xi)}\right\rangle dx \\
&\quad + \frac{4(n-1)}{n-2}\int_{B_{2\alpha}} u_{(0,\xi)}^2 |\nabla\eta|^2\, dx \\
&= I + II + III.
\end{aligned}$$

We estimate I, II, and III. Note

$$\begin{aligned}
I &= \frac{4(n-1)}{n-2}\int_{B_{2\alpha}} \eta^2 \left|\nabla u_{(0,\xi)}\right|^2 dx \\
&\leq \frac{4(n-1)}{n-2}\int_{\mathbb{R}^n} \left|\nabla u_{(0,\xi)}\right|^2 dx \\
&= \Lambda \left(\int_{\mathbb{R}^n} u_{(0,\xi)}^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}.
\end{aligned}$$

To estimate II and III, we need

$$\begin{aligned}
\int_{B_{2\alpha}\setminus B_\alpha} \left|\nabla u_{(0,\xi)}\right|^2 dx &\leq c\varepsilon^{n-2}\int_\alpha^{2\alpha} \frac{r^{n-1}}{r^{2n-2}} dr = c\varepsilon^{n-2} \\
&\leq \frac{c\varepsilon^{n-2}}{\alpha^{n-2}}
\end{aligned}$$

and

$$\int_{B_{2\alpha}\setminus B_\alpha} u_{(0,\varepsilon)}^{\frac{2n}{n-2}} dx \leq c\frac{\varepsilon^n}{\alpha^n}.$$

So

$$\begin{aligned}
II &= \frac{4(n-1)}{n-2}\int_{B_{2\alpha}\setminus B_\alpha} 2\eta u_{(0,\varepsilon)} \left\langle \nabla u_{(0,\varepsilon)}, \nabla\eta\right\rangle dx \\
&\leq \frac{c}{\alpha}\left(\int_{B_{2\alpha}\setminus B_\alpha} |u(0,\varepsilon)|^2\right)^{\frac{1}{2}} \left(\int_{B_{2\alpha}} \left|\nabla u_{(0,\varepsilon)}\right|^2\right)^{\frac{1}{2}} \\
&\leq \frac{c}{\alpha}\left[\left(\int_{B_{2\alpha}\setminus B_\alpha} |u(0,\varepsilon)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \left(\int_{B_{2\alpha}\setminus B_\alpha} dx\right)^{\frac{2}{n}}\right]^{\frac{1}{2}} \left(\int_{B_{2\alpha}\setminus B_\alpha} |\nabla u|^2\, dx\right)^{\frac{1}{2}} \\
&\leq \frac{c}{\alpha}\left[\left(\frac{\varepsilon}{\alpha}\right)^{n-2}\alpha^2\right]^{\frac{1}{2}} \left(\frac{\varepsilon}{\alpha}\right)^{\frac{n-2}{2}} \\
&\leq c\left(\frac{\varepsilon}{\alpha}\right)^{n-2}.
\end{aligned}$$

Similarly,

$$III = \frac{4(n-1)}{n-2}\int_{B_{2\alpha}\setminus B_\alpha} u_{(0,\xi)}^2 |\nabla\eta|^2\, dx$$

$$\leq \frac{c}{\alpha^2}\left(\int_{B_{2\alpha}\setminus B_\alpha} u_{(0,\varepsilon)}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}\left(\int_{B_{2\alpha}\setminus B_\alpha}\right)^{\frac{2}{n}}$$
$$\leq c\left(\frac{\varepsilon}{\alpha}\right)^{n-2}.$$

Thus, we get

$$\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n}|\nabla\varphi|^2\,dx \leq \left(\int_{\mathbb{R}^n} u_{(0,\xi)}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} + \left(\frac{\varepsilon}{\alpha}\right)^{n-2}.$$

Since $\varphi = \eta u_{(0,\varepsilon)}$ and φ is supported in $B_{2\alpha}$ with $\varphi = 1$ on B_α, we have

$$\left|\left(\int_{\mathbb{R}^n} u_{(0,\xi)}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} - \left(\int_{B_{2\alpha}} \varphi^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\right|$$
$$\leq c\left(\int_{\mathbb{R}^n\setminus B_\alpha}\left|u_{(0,\varepsilon)}-\varphi\right|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}.$$

Since $\varphi_{(0,\varepsilon)} \geq \varphi \geq 0$, we have

$$\leq c\left(\int_{\mathbb{R}^n\setminus B_\alpha} u_{(0,\varepsilon)}^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}$$
$$\leq c\left(\frac{\varepsilon}{\alpha}\right)^{n-2}.$$

Thus,

$$\frac{\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n}|\nabla\varphi|^2\,dx}{\left(\int_{B_{2\alpha}}\varphi^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}} \leq \Lambda + c\left(\frac{\varepsilon}{\alpha}\right)^{n-2}.$$

Now for a compact Riemannian manifold (M,g), choose any point $x_0 \in M$. Then we can consider the geodesic normal neighborhood U of x_0. Then U can be parametrized by $(\xi_1,\ldots,\xi_n)$ such that

$$g\left(\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j}\right) = \delta_{ij} \quad \text{in } U$$
$$\Gamma_{ij}^k(x_0) = 0.$$

In this neighborhood,

$$|\nabla_g\varphi|_g^2 = |\nabla\varphi|^2.$$

Similarly,

$$dV_g = d\xi$$

since $g\left(\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\right) = \delta_{ij}$ in U. Therefore, we can construct a C^∞-function φ compactly supported in $U = B_{2\alpha}$ such that

$$\frac{\frac{4(n-1)}{n-2}\int_U|\nabla_g\varphi|_g^2\,dV_g}{\left(\int_U\varphi^{\frac{2n}{n-2}}dV_g\right)^{\frac{n-2}{n}}} \leq \Lambda + c\left(\frac{\varepsilon}{\alpha}\right)^{n-2}.$$

Extend φ to a C^∞-function in M by letting $\varphi = 0$ in $M \setminus U$. Then using Hölder's inequality,

$$\begin{aligned} E_g(\varphi) &= \frac{\frac{4(n-1)}{n-2}\int_M |\nabla_g \varphi|^2 \, dV_g + \int_M R_g \varphi^2 dV_g}{\left(\int_U \varphi^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \\ &\leq \frac{4(n-1)}{n-2} \frac{\int_M |\nabla_g \varphi|^2 \, dV_g}{\left(\int_U \varphi^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} dx + \max_M |R_g| \frac{\left(\int_M \varphi^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} \left(\int_U dV_g\right)^{\frac{2}{n}}}{\left(\int_U \varphi^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \\ &\leq \Lambda + c\left(\frac{\varepsilon}{\alpha}\right)^{n-2} + c\alpha^2. \end{aligned}$$

First, choose α very small since we can choose the radius of normal ball very small. Next, we pass $\varepsilon \to 0$. Thus, $E_\varphi(\varphi)$ is arbitrary closed to Λ. Thus,

$$Y(M, g) \leq \Lambda = Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

This completes the proof of (i). □

To prove (ii), we have the following difficulty. We have the following Sobolev embedding, where $W^{1,2}(M) \hookrightarrow L^p(M)$, where $1 < p \leq \frac{2n}{n-2}$. This embedding is compact when $1 < p < \frac{2n}{n-2}$ and is only continuous when $p = \frac{2n}{n-2}$.

The strategy of Yamabe is to consider the subcritical equation of Yamabe equation: for $1 < p < \frac{2n}{n-2}$, consider

$$-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u = R_g u^{p-1} \tag{2.7}$$

with R_g =constant.

(a) We can find a solution u_p for (2.7).

(b) Conclude that $u_p \to u$ as $p \to \frac{2n}{n-2}$ and u satisfies the Yamabe equation.

Yamabe succedded in doing step (a) but failed in doing step (b). Aubin-Trudinger proved (b) under the condition $Y(M, g) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$.

We first do Step (a). Define

$$E_p(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla_{g_0} u|^2 + R_{g_0} u^2\right) dV_{g_0}}{\left(\int_M u^p dV_{g_0}\right)^{\frac{2}{p}}}$$

for $1 < p \leq \frac{2n}{n-2}$. When $p = \frac{2n}{n-2}$, E_p is the original Yamabe energy. Define

$$Y_p(M, g_0) = \inf\left\{E_p(u) : 0 \neq u \in W^{1,2}(M)\right\}.$$

Following the proof of Proposition 1.8, we can show that if u is a critical point of E_p, then u satisfies (2.7). In particular, if u is a minimizer in the sense that

$$E_p(u) = Y_p(M, g_0),$$

then u satisfies (2.7).

Following the proof of Proposition 1.15, we obtain the following: Let $\lambda_1(g_0)$ be the first eigenvalue of L_{g_0}. If $\lambda_1(g_0) < 0$, then $Y_p(M, g_0) < 0$. If $\lambda_1(g_0) \geq 0$, then $Y_p(M, g_0) \geq 0$.

Lemma 2.15.

(i) $\limsup_{p\to \frac{2n}{n-2}} Y_p(M,g_0) \le Y(M,g_0)$.

(ii) *If* $Y_p(M,g_0)\ge 0$*, then* $Y_p(M,g_0)\to Y(M,g_0)$ *as* $p\to \frac{2n}{n-2}$.

Proof. (i) Note that

$$Y(M,g_0) = Y_{\frac{2n}{n-2}}(M,g_0).$$

Then there exists $0<u_i\in C^\infty(M)$ such that

$$E(u_i) = \frac{\int_M u_i L_{g_0} u_i dV_{g_0}}{\left(\int_M |u_i|^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}} \le Y(M,g_0) + \frac{1}{i}.$$

Consider

$$E_p(u_i) = \frac{\int_M u_i L_{g_0} u_i dV_{g_0}}{\left(\int_M |u_i|^p dV_{g_0}\right)^{\frac{2}{p}}}.$$

Since $p\mapsto \left(\int_M |u_i|^p dV_{g_0}\right)^{\frac{1}{p}}$ is continuous,

$$\lim_{p\to\frac{2n}{n-2}} E_p(u_i) = E(u_i) \le Y(M,g_0)+\frac{1}{i}.$$

Thus,

$$\limsup_{p\to\frac{2n}{n-2}} Y_p(M,g_0) \le \limsup_{p\to\frac{2n}{n-2}} E_p(u_i) \le Y(M,g_0)+\frac{1}{i}.$$

Letting $i\to\infty$, we are done.

(ii) If $Y_p(M,g_0)\ge 0$, then $E_p(u)\ge 0$ for all $0<u\in C^\infty(M)$. Then by Hölder's inequality,

$$\begin{aligned}
E(u) &= E_p(u)\frac{\left(\int_M |u|^p dV_{g_0}\right)^{\frac{2}{p}}}{\left(\int_M |u|^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}} \\
&\le E_p(u)\frac{\left(\int_M |u|^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}}{\left(\int_M |u|^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}} \mathrm{Vol}(M,g_0)^{\frac{2}{p}\left(1-p\frac{n-2}{2n}\right)}. \\
&= E_p(u)\,\mathrm{Vol}(M,g_0)^{\frac{2}{p}\left(1-p\frac{n-2}{2n}\right)\cdot}.
\end{aligned}$$

Taking infimum over $0<u\in C^\infty(M)$, we obtain

$$Y(M,g_0) \le Y_p(M,g_9)\,\mathrm{Vol}(M,g_0)^{\frac{2}{p}\left(1-p\frac{n-2}{2n}\right)}.$$

Now taking liminf , we have

$$Y(M,g_0) \le \liminf_{p\to\infty} Y_p(M,g_0).$$

Hence combinig this with (i), this completes the proof. ☐

Lemma 2.16. *For any* $2 < p < \frac{2n}{n-2}$, *there exists* $0 < u_p \in C^\infty(M)$ *such that*

$$\|u_p\|_{L^p(M,g_0)} = 1,$$

$$E_p(u_p) = Y(M, g_0),$$

and

$$L_{g_0} u_p = Y_p(M, g_0)\, u_p^{p-1}. \tag{2.8}$$

Proof. Choose a sequence $\{u_i\} \subset W^{1,2}(M) \setminus \{0\}$ such that $E_p(u_i) \to Y_p(M, g_0)$ as $i \to \infty$. Since $E_p(|u_i|) \leq E_p(u_i)$, we can assume that u_i are nonnegative. Since Yamabe energy is scaling invariant, by rescaling, we may assume that $\|u_i\|_{L^p(M,g_0)} = 1$. First we show that $\{u_i\}$ is a bounded sequence in $W^{1,2}(M)$. By $E_p(u_i) \to Y_p(M, g_0)$ as $i \to \infty$ and $\|u_i\|_{L^p} = 1$ for all i, we have

$$\frac{4(n-1)}{n-2} \|\nabla u_i\|_{L^2}^2 - \max_M |R_{g_0}| \, \|u_i\|_{L^2} \leq C$$

for some constant C independent of i. Thus, by Hölder's inequality,

$$\|\nabla u_i\|_{L^2}^2 \leq c_1 + c_2(M, n, p)\, \|u_i\|_{L^p}^{\frac{2}{p}} \leq C_1,$$

where C_1 is a constant independent of i.

By the weak compactness result of $W^{1,2}(M)$, there exists a subsequence of $\{u_i\}$ which is still denoted by $\{u_i\}$ such that u_i converges to u weakly in $W^{1,2}(M)$ as $i \to \infty$, i.e.,

$$\lim_{i\to\infty} \int_M \left(\langle \nabla_{g_0} u_i, \nabla_{g_0} f\rangle + u_i f\right) dV_{g_0} = \int_M \left(\langle \nabla_{g_0} u, \nabla_{g_0} f\rangle + u f\right) dV_{g_0} \tag{2.9}$$

for any $f \in W^{1,2}(M)$. By the Rellich-Kondrachov theorem, we know that $W^{1,2}(M)$ is compactly embedded in $L^p(M)$. Thus, (up to subsequence) $u_i \to u$ strongly in $L^p(M)$ as $i \to \infty$. That is,

$$\lim_{i\to\infty} \|u_i - u\|_{L^p} = 0. \tag{2.10}$$

This implies

$$\lim_{i\to\infty} \|u_i\|_{L^p} = \|u\|_{L^p}.. \tag{2.11}$$

Therefore,

$$\begin{aligned} \int_M (u_i - u)\, f dV_{g_0} &\leq \|u_i - u\|_{L^2(M)} \|f\|_{L^2(M)} \\ &\leq c \|u_i - u\|_{L^p(M)} \|f\|_{L^2(M)} \to 0 \end{aligned}$$

as $i \to \infty$ by (2.10). Thus, combining this with (2.9), we have

$$\lim_{i\to\infty} \int_M \langle \nabla_{g_0} u_i, \nabla_{g_0} f\rangle\, dV_{g_0} = \int_M \langle \nabla_{g_0} u, \nabla_{g_0} f\rangle\, dV_{g_0} \tag{2.12}$$

for $f \in W^{1,2}(M)$.

Put $f = u_i - u$ in (2.12). Then

$$\lim_{i\to\infty} \int_M |\nabla_{g_i}(u_i - u)|^2\, dV_{g_0} = 0,$$

i.e.,

$$\lim_{i\to\infty} \int_M |\nabla_{g_0} u_i|^2 - 2\int_M \langle \nabla_{g_0} u_i, \nabla_{g_0} u\rangle\, dV_{g_0} + \int_M |\nabla_{g_0} u|^2\, dx = 0.$$

Put $f = u$ in (2.12). Then

$$\lim_{i\to\infty}\int_M \langle \nabla_{g_0} u_i, \nabla u\rangle \, dV_{g_0} = \int_M |\nabla_{g_0} u|^2 \, dV_{g_0}.$$

Thus,

$$\lim_{i\to\infty}\int_M |\nabla_{g_0} u_i|^2 \, dV_{g_0} = \int_M |\nabla_{g_0} u|^2 \, dV_{g_0}. \tag{2.13}$$

Since $\{u_i\}$ converges weakly in $W^{1,2}(M)$, $\{u_i\}$ is bounded in $W^{1,2}(M)$. By Hölder's inequality,

$$\begin{aligned}\left|\int_M R_{g_0} u_i^2 dV_{g_0} - \int_M R_{g_0} u^2 dV_{g_0}\right| &\le \max_M |R_{g_0}| \int_M |u_i - u|\,|u_i + u| \, dV_{g_0} \\ &\le C\left(\int_M |u_i - u|^2 \, dV_{g_0}\right)^{\frac{1}{2}} \\ &\le C \|u_i - u\|_{L^p(M)}.\end{aligned} \tag{2.14}$$

Combining (2.11), (2.13) and (2.14), we have

$$Y_p(M, g_0) = E_p(u) = \lim_{i\to\infty} E_p(u_i) = E_p(u) \ge Y_p(M, g_0).$$

Thus, $E_p(u) = Y_p(M, g_0)$. By (2.11), $\|u\|_{L^p} = 1$. As a minimizer, u satisfies $L_{g_0} u = Y_p(M, g_0)\, u^{p-1}$ in the weak sense.

It remains to show that $u_p > 0$ and $u_p \in C^\infty(M)$. We can do this by using bootstrap argument, which means the following: for any $p - 1 < q < \frac{2n}{n-2}$,

$$L_g u_p = Y_p(M, g_0)\, u_p^{p-1} \in L^{\frac{q}{p-1}}(M, g_0).$$

Hence by L^p estimates of conformal Laplacian, $u_p \in W^{2,\frac{q}{p-1}}(M, g_0)$. Hence by Sobolev embedding theorem, $u_p \in L^s(M, g_0)$, where

$$s = \frac{n\left(\frac{q}{p-1}\right)}{n - 2\left(\frac{q}{p-1}\right)} > \frac{q}{p-1}.$$

Therefore, we get

$$L_g u_p = Y_p(M, g_0)\, u_p^{p-1} \in L^{\frac{s}{p-1}}(M, g_0)$$

with $s > q$. Hence by L^p estimates of conformal Laplacian, $u \in W^{2,\frac{s}{p-1}}(M, g_0)$. Repeating the above procedure, we get $u \in W^{2,q}(M, g_0)$ for all $q > 1$. Hence by Morrey embedding theorem, $u \in C^\alpha(M, g_0)$ for some α, where C^α denotes the space of Hölder's continuous.

Since $L_{g_0} u_p = Y_p(M, g_0)\, u_p^{p-1} \in C^\alpha(M, g_0)$ for any $0 < \alpha < 1$, by the Schauder estimates, $u_p \in C^{2,\alpha}(M, g_0)$. Since $L_{g_0} u_p = Y_p(M, g_0)\, u_p^{p-1} \in C^{2,\alpha}$, derivatives of u also satisfies some ellliptic equation. So we can apply the same techniques to conclude that all derivatives of u_p is $C^{2,\alpha}$. Continue this process to conclude that $u_p \in C^\infty(M)$.

Recall u_p is the limit of sequence of nonnegative functions. By maximum principle, $u_p = 0$ or $u_p > 0$. Since $\|u_p\|_{L^p} = 1$, $u_p > 0$. This completes the proof of Lemma. □

Let us start Step (b) of Yamabe's strategy. Letting $p \to \frac{2n}{n-2}$ and we want to conclude that $u_p \to u$ which satisfies

$$L_{g_0} u = Y(M, g_0)\, u^{\frac{n+2}{n-2}}.$$

Yamabe claimed that he can do this. But this is false in general. (See Example 2.18)

However, Aubin-Trudinger proved that the claim of Yamabe is true under the technical assumption.

Lemma 2.17 (Aubin-Trudinger)**.** *Let u_p be a solution constructed in Lemma.. If $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$, then there exist constants $s_0 < \frac{2n}{n-2}$ and $r > \frac{2n}{n-2}$ and $C > 0$ such that*

$$\|u_p\|_{L^r(M,g_0)} \le C$$

for all $s_0 \le p < \frac{2n}{n-2}$, where

$$L_{g_0} u_p = Y(M, g_0)\, u_p^{p-1}. \tag{2.15}$$

Proof. By rescaling g_0, we may assume that $\mathrm{Vol}(M, g_0) = 1$. Let $\delta > 0$. Multiply $u_p^{1+2\delta}$ to (2.15) and integrate it. Then integration by part gives

$$\begin{aligned} Y_p(M, g_0) \int_M u_p^{p+2\delta} dV_{g_0} &= \int_M u_p^{1+2\delta} L_{g_0} u_p dV_{g_0} \\ &= -\frac{4(n-1)}{n-2} \int_M u_p^{1+2\delta} \triangle_{g_0} u_p + \int_M R_{g_0} u_p^{2+2\delta} dV_{g_0} \\ &= \frac{4(n-1)}{n-2} (1+2\delta) \int_M u_p^{2\delta} |\nabla_{g_0} u_p|^2 dV_{g_0} + \int_M R_{g_0} u_p^{2+2\delta} dV_{g_0}. \end{aligned} \tag{2.16}$$

If we set $w = u_p^{1+\delta}$, we have $w^2 = u_p^{2+2\delta}$ and $|\nabla_{g_0} w|^2 = (1+\delta)^2 u_p^{2\delta} |\nabla_{g_0} u_p|^2$. Put these into (2.16), we get

$$\frac{4(n-1)}{n-2} \frac{1+2\delta}{(1+\delta)^2} \int_M |\nabla_{g_0} w|^2 dV_{g_0} = \int_M \left(Y_p(M, g_0) w^2 u_p^{p-2} - R_{g_0} w^2\right) dV_{g_0}. \tag{2.17}$$

Now by the Hölder's inequality, the sharp Sobolev inequality and (2.17), we get

$$\begin{aligned} \|v\|^2_{L^{\frac{2n}{n-2}}} &\le (1+\varepsilon) \frac{1}{\Lambda} \int_M \frac{4(n-1)}{n-2} |\nabla_{g_0} w|^2 dV_{g_0} + C_\varepsilon \int_M w^2 dV_{g_0} \qquad (2.18) \\ &\le \frac{1+\varepsilon}{\Lambda} \frac{(1+\delta)^2}{1+2\delta} \int_M \left(Y_p(M, g_0) w^2 u_p^{p-2} - R_{g_0} w^2\right) dV_{g_0} + C_\varepsilon \int_M w^2 dV_{g_0} \\ &\le \frac{1+\varepsilon}{\Lambda} \frac{(1+\delta)^2}{1+2\delta} Y_p(M, g_0) \left(\int_M w^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}} \left(\int_M u_p^{(p-2)\frac{n}{2}} dV_{g_0}\right)^{\frac{2}{n}} + C_\varepsilon \int_M w^2 dV_{g_0}. \end{aligned}$$

Note that $p < \frac{2n}{n-2}$ if and only if $(p-2)\frac{n}{2} < p$. So by Hölder's inequality

$$\int_M u_p^{(p-2)\frac{n}{2}} dV_{g_0} \le 1$$

since $\|u_p\|_{L^p} = 1$ and $\mathrm{Vol}(M, g_0) = 1$.

If $Y_p(M, g_0) \ge 0$, then by Lemma 2.15, $Y_p(M, g_0) \to Y(M, g_0)$ as $p \to \frac{2n}{n-2}$. Since we have assumed

$$\frac{Y(M, g_0)}{Y(\mathbb{S}^n, g_0)} = \frac{Y(M, g_0)}{\Lambda} < 1,$$

we can choose δ and ε sufficiently small so that

$$\gamma = (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{Y_p(M, g_0)}{\Lambda} < 1.$$

So we get

$$(1-\gamma) \|w\|^2_{L^{\frac{2n}{n-2}}} \le c \|w\|^2_{L^2}.$$

This is also true for the case $Y_p(M, g_0)$ is negative. We can conclude that

$$\|w\|^2_{L^{\frac{2n}{n-2}}} \le C\,\|w\|^2_{L^2}\,.$$

Here the constant C is independent of p. Choose δ sufficiently small so that $2+2\delta < p$. Then by Hölder's inequality, we get

$$\|w\|^2_{L^2} = \int_M w^2 dV_{g_0} = \int_M u_p^{2+2\delta} dV_{g_0} \le 1.$$

Note that

$$\|w\|^2_{L^{\frac{2n}{n-2}}} = \left(\int_M u_p^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}}.$$

This completes the proof of Lemma 2.17. □

Based on Lemma 2.17, we can conclude that Step (b) works under the condition $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$. Consisder the solution u_p of $L_{g_0} u_p = Y_p(M, g_0)\, u_p^{p-1}$. By Lemma 2.17, $\{u_p\}$ are uniformly bounded in L^r for some $r > \frac{2n}{n-2}$.

For each fixed $s_0 \le p < \frac{2n}{n-2}$, $L_{g_0} u_p = Y(M, g_0)\, u_p^{p-1} \in L^{\frac{r}{p-1}}(M, g_0)$. So $u_p \in W^{2,\frac{r}{p-1}}(M, g_0)$. By the Sobolev embedding theorem, $\|u_p\|_{L^{r_1}} \le C$ for some $r_1 > r$. Here the constant C is independent of p. So $\|u_p\|_{W^{2,q}(M)} \le C$ for all q independent of p. Thus, by Schauder estimate, $\|u_p\|_{C^{2,\alpha}(M)} \le C$ for all α. Here the constant C is independent of p.

Thus, by Arzelá-Ascoli theorem, there exists a convergent subsequence of $\{u_p\}$ which converges to a function $u \in C^2(M)$ as $p \to \frac{2n}{n-2}$ and u satisfies

$$L_{g_0} u = Y u^{\frac{n+2}{n-2}}, \tag{2.19}$$

where

$$Y = \lim_{p \to \frac{2n}{n-2}} Y_p(M, g_0) \begin{cases} = Y(M, g_0) & \text{if } Y(M, g_0) > 0 \\ \le Y(M, g_0) & \text{if } Y(M, g_0) \le 0. \end{cases}$$

Now multiply (2.19) and

$$Y = E(u) \ge Y(M, g_0)$$

by definition. This completes the proof of Theorem 2.1.

Example 2.18. The crucial step of the above proof was $\|u_p\|_{C^{2,\alpha}(M)}$ is uniformly bounded for any $p < \frac{2n}{n-2}$. Yamabe claimed this. But this is not true for the case when $M = \mathbb{S}^n$. Recall the identity (2.2):

$$(\pi^{-1})^* g_0 = \frac{4}{(1+|x|^2)} ds^2.$$

Set $u_1(x) = \frac{1}{(1+|x|^2)^{(n-2)/2}}$. Then we write

$$(\pi^{-1})^* g_0 = 4 u_1^{\frac{2n}{n-2}-2} ds^2.$$

Using the stereographic projection, we can find all conformal diffeomorphisms on sphere. This diffeomorphism is generated by rotation, $\pi^{-1}\tau_v\pi$, and $\pi^{-1}\delta_\alpha\pi$. Here τ_v and δ_α are

$$\tau_v(x) = x - v, \quad \delta_\alpha(x) = \alpha^{-1} x$$

for $v \in \mathbb{R}^n$ and $\alpha > 0$. In particular,

$$\delta^*(\pi^{-1})^* g_0 = 4 u_\alpha^{\frac{2n}{n-2}-2} ds^2,$$

where $u_\alpha(x) = \left(\frac{\alpha}{|x|^2+\alpha^2}\right)^{\frac{2-n}{2}}$. Note that $(\pi^{-1}\delta_\alpha\pi)^* g_0$ is a solution of Yamabe equation for any $\alpha > 0$ but not uniformly bounded.

2.2 Yamabe problem under Aubin's assumption

We have proved Aubin-Trudinger theorem. To solve the Yamabe problem, we just need to show that

$$Y(M,g) < Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

So by definition, if we can find $0 \neq u \in C^\infty(M)$ such that $E(u) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$, then

$$Y(M,g) = \inf\{E(u)\} < Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

We know that $Y(M,g) \leq Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ by considering bump function. But it is not that easy. For some cases, we can do this. Historically, this was done by Aubin in 1976 when (M,g) is not locally conformally flat and $\dim M \geq 6$. In 1984, Schoen proved this when (M,g) is locally conformally flat or $3 \leq \dim M \leq 5$ by using positive mass theorem.

Today, we consider Aubin's theorem. To do this, we introduce the concept 'conformal normal coordinates'. Recall, the geodesic normal coordinates is given by: for any $p \in (M, g_0)$, there is a neighborhood U of p such that $g_0(X_i, X_j) = \delta_{ij}$ in U and $\Gamma_{ij}^k(p) = 0$ at p.We can do something similarly in conformal geometry. This is called *conformal normal coordinates*, which was introduced by Robin Graham.

Theorem 2.19. *Let (M, g_0) be a Riemannian manifold. Fix $p \in M$. For each $N \geq 2$, there exists $g \in [g_0]$ such that*

(i) $\det(g_{ij}) = 1 + O(r^N)$, *where $r = |x|$ in the normal coordinates at p.*

(ii) *In these coordinates, if $N \geq 5$, then the scalar curvature of g satisfies*

$$R_g = O(r^2), \quad R_g(p) = 0, \quad \text{and} \quad \triangle_g R_g = -\frac{1}{6}|W|^2 \quad \text{at } p.$$

Here

$$|W|^2 = g^{ii'} g^{jj'} g^{kk'} g^{ll'} W_{ijkl} W_{i'j'k'l'}.$$

Basically, the proof is nothing but linear algebra. See Lee and Parker or Scheon and Yau. Assuming this, we can prove the following theorem.

Theorem 2.20 (Aubin)**.** *If (M, g_0) is not locally conformally flat and* $\dim M \geq 6$*, then*

$$Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

Proof. Let $\{X_i\}$ be a conformal normal coordinates in a neighborhood of $p \in M$. We have constructed the a C^∞-function φ compactly supported in U such that

$$\frac{\frac{4(n-1)}{n-2}\int_U |\nabla_g \varphi|^2 \, dV_g}{\left(\int_U \varphi^{\frac{2n}{n-2}} \, dV_g\right)^{\frac{n-2}{n}}} \leq \Lambda + \left(\frac{\varepsilon}{\alpha}\right)^{n-2},$$

where $\varphi = \eta u_{(0,\varepsilon)}$ and

$$\eta = \begin{cases} 1 & \text{in } B_\alpha \\ 0 & \text{outside } B_{2\alpha}. \end{cases}$$

The idea of this theorem is to consider scalar curvature terms. Actually, this motivates us to study the conformal normal coordinates. By considering Taylor's series of R_g expanding at p in coordinates $\{x_i\}$,

$$\int_U R_g \varphi^2 dV_g$$
$$= \int_{B_{2\alpha}} \left[R_g(p) + \sum_{i=1}^n \frac{\partial}{\partial x_i} R_g(p) x_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} R_g(p) x_i x_j + O\left(r^3\right) \right] \varphi^2 dx.$$

Using the co-area formula, we have

$$= \int_{B_{2\alpha}} \left[R_g(p) + \sum_{i=1}^n \frac{\partial}{\partial x_i} R_g(p) x_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 R_g}{\partial x_i^2}(p) x_i^2 + O\left(r^3\right) \right] \varphi^2 dx$$
$$= \int_0^{2\alpha} \int_{\partial B_r} \left[R_g(p) + \sum_{i=1}^n \frac{\partial}{\partial x_i} R_g(p) x_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 R_g}{\partial x_i^2}(p) x_i^2 + O\left(r^3\right) \right] \varphi^2 d\sigma dr.$$

Since φ is a radial function and x_i is a odd function with respect to the origin, we have

$$\int_0^{2\alpha} \left(\int_{\partial B_r} x_i d\sigma \right) \varphi^2 dr = 0.$$

Similarly, we have

$$\int_0^{2\alpha} \left(\int_{\partial B_r} x_i x_j d\sigma \right) \varphi^2 dr = 0$$

if $i \neq j$.

For fixed $r > 0$, for any $1 \leq i \leq n$, change of variable gives

$$\int_{\partial B_r} x_i^2 d\sigma = \int_{\mathbb{S}^{n-1}} (r y_i)^2 r^{n-1} d\sigma = r^{n+1} \int_{\mathbb{S}^{n-1}} y_i^2 d\sigma.$$

Since the surface measure on $\mathbb{S}^{n-1}$ is invariant under the orthogonal transformation, we see that

$$\int_{\mathbb{S}^{n-1}} y_i^2 d\sigma = \int_{\mathbb{S}^{n-1}} y_1^2 d\sigma$$

for any $1 \leq i \leq n$.

So

$$\int_{\mathbb{S}^{n-1}} y_i^2 d\sigma = \frac{\omega_n}{n}.$$

Thus,

$$\int_U R_g \varphi^2 dV_g = \int_{B_{2\alpha}} \left(R_g(p) + \frac{1}{2} \triangle_g R_g(p) \frac{r^2}{n} + O\left(r^3\right) \right) \varphi^2 dx$$
$$= -c\left|W(p)\right|^2 \int_{B_{2\alpha}} r^2 \eta^2 u_{(0,\varepsilon)}^2 dx + \int_{B_{2\alpha}} O\left(r^3\right) \eta^2 u_{(0,\varepsilon)}^2 dx.$$

The following elementary lemma helps us to estimate this.

Lemma 2.21. *Suppose $k > -n$ and $\alpha > 0$ is fixed. Then, as $\varepsilon \to 0$,*

$$I(\varepsilon) = \int_0^\alpha r^k u_{(0,\varepsilon)}^2 r^{n-1} dr$$

is bounded above and below by positive multiple of

$$\begin{cases} \varepsilon^{k+2} & \textit{if } n > k+4 \\ \varepsilon^{k+2} \log\left(\frac{1}{\varepsilon}\right) & \textit{if } n = k+4 \\ \varepsilon^{n-2} & \textit{if } n < k+4. \end{cases}$$

Proof. By change of variables, we have

$$\begin{aligned} I(\varepsilon) &= \int_0^{\alpha} r^k \left(\frac{\varepsilon}{\varepsilon^2 + r^2}\right)^{n-2} r^{n-1} dr \\ &= \int_0^{\frac{\alpha}{\varepsilon}} (\varepsilon\tilde{r})^k \left(\frac{\varepsilon}{\varepsilon^2 + \varepsilon^2\tilde{r}^2}\right)^{n-2} (\varepsilon\tilde{r})^{n-1} \varepsilon d\tilde{r} \\ &= \varepsilon^{k+2} \int_0^{\frac{\alpha}{\varepsilon}} \frac{\tilde{r}^{k+n-1}}{(1+\tilde{r}^2)^{n-2}} d\tilde{r}. \end{aligned}$$

If ε is sufficiently small, $\frac{\alpha}{\varepsilon} \geq 1$. So

$$\begin{aligned} I(\varepsilon) &\sim \varepsilon^{k+2} \left(\int_0^1 \frac{\tilde{r}^{k+n-1}}{(1+\tilde{r}^2)^{n-2}} d\tilde{r} + \int_1^{\frac{\alpha}{\varepsilon}} \frac{\tilde{r}^{k+n-1}}{\tilde{r}^{2(n-2)}} d\tilde{r}\right) \\ &= \varepsilon^{k+2} \left(c + \int_1^{\frac{\alpha}{\varepsilon}} \tilde{r}^{k+3-n} d\tilde{r}\right). \end{aligned}$$

If $n > k+4$, then $k+3-n < -1$. So $\int_1^{\frac{\alpha}{\varepsilon}} \tilde{r}^{k+3-n} d\tilde{r} \sim C$ for some constant C. If $n = k+4$, then

$$\int_1^{\frac{\alpha}{\varepsilon}} \tilde{r}^{k+3-n} d\tilde{r} = c \log\left(\frac{\alpha}{\varepsilon}\right).$$

If $n < k+4$, then

$$\int_1^{\frac{\alpha}{\varepsilon}} \tilde{r}^{k+3-n} d\tilde{r} = c \left[\tilde{r}^{k+4-n}\right]_1^{\frac{\alpha}{\varepsilon}} = c\left(\left(\frac{\alpha}{\varepsilon}\right)^{k+4-n} - 1\right) \sim \frac{c}{\varepsilon^{k+4-n}}.$$

Thus, we are done. □

By this Lemma with $k = 2$, we have

$$\begin{aligned} \int_{B_{2\alpha}} r^2 \eta^2 u_{(0,\varepsilon)}^2 dx &\geq \int_{B_\alpha} r^2 u_{(0,\varepsilon)}^2 dx \\ &= \omega_{n-1} \int_0^{\alpha} r^2 u_{(0,\varepsilon)}^2 r^{n-1} dr \\ &\geq \begin{cases} c\varepsilon^4 & \text{if } n > 6, \\ c\varepsilon^4 \log\left(\frac{1}{\varepsilon}\right) & \text{if } n = 6. \end{cases} \end{aligned}$$

So

$$\begin{aligned} &-\frac{1}{12n} |W(p)|^2 \int_{B_{2\alpha}} r^2 \eta^2 u_{(0,\varepsilon)}^2 dx \\ &\leq \begin{cases} -c|W(p)|^2 \varepsilon^4 & \text{if } n > 6, \\ -c|W(p)|^2 \varepsilon^4 \log\left(\frac{1}{\varepsilon}\right) & \text{if } n = 6. \end{cases} \end{aligned}$$

Similarly, by the Lemma with $k = 3$,

$$\begin{aligned} &\int_{B_{2\alpha}} O(r^3) \eta^2 u_{(0,\varepsilon)}^2 dx. \\ &\leq c \int_{B_{2\alpha}} r^3 u_{(0,\varepsilon)}^2 dx \end{aligned}$$

$$
\begin{aligned}
&= c\omega_{n-1}\int_0^{2\alpha} r^3 u_{(0,\varepsilon)}^2 r^{n-1}dr\\
&\le \begin{cases} c\varepsilon^5 & \text{if } n>7,\\ c\varepsilon^5\log\left(\frac{1}{\varepsilon}\right) & \text{if } n=7,\\ c\varepsilon^4 & \text{if } n=6.\end{cases}
\end{aligned}
$$

Thus, combining these two estimates, we obtain

$$
\begin{aligned}
&-\frac{1}{12n}\left|W(p)\right|^2\int_{B_{2\alpha}} r^2\eta^2 u_{(0,\varepsilon)}^2dx+\int_{B_{2\alpha}} O\left(r^3\right)\eta^2u_{(0,\varepsilon)}^2dx\\
&\le \begin{cases} -c\left|W(p)\right|^2\varepsilon^4+C\varepsilon^5\log\left(\frac{1}{\varepsilon}\right) & \text{if } n>6,\\ -c\left|W(p)\right|^2\varepsilon^4\log\left(\frac{1}{\varepsilon}\right)+C\varepsilon^4 & \text{if } n=6.\end{cases}
\end{aligned}
$$

Since (M, g_0) is not locally conformally flat and $n\ge 6$, there exists a point $p\in M$ such that $\left|W(p)\right|^2>0$. For this point p, we have

$$
\int_U R_g\varphi^2dV_g\le \begin{cases} -c\left|W(p)\right|^2\varepsilon^4+C\varepsilon^5\log\left(\frac{1}{\varepsilon}\right) & \text{if } n>6,\\ -c\left|W(p)\right|^2\varepsilon^4\log\left(\frac{1}{\varepsilon}\right)+C\varepsilon^4 & \text{if } n=6.\end{cases}
$$

if ε is sufficiently small.

Thus,

$$
\begin{aligned}
E(\varphi) &= \frac{\frac{4(n-1)}{n-2}\int_U\left|\nabla_g\varphi\right|^2dV_g+\int_U R_g\varphi^2dV_g}{\left(\int_U\varphi^{\frac{2n}{n-2}}dV_g\right)^{\frac{n-2}{n}}}\\
&\le \Lambda+\left(\frac{\varepsilon}{\alpha}\right)^{n-2}-\begin{cases} c\left|W(p)\right|^2\varepsilon^4-C\varepsilon^5\log\left(\frac{1}{\varepsilon}\right) & \text{if } n>6,\\ c\left|W(p)\right|^2\varepsilon^4\log\left(\frac{1}{\varepsilon}\right)-C\varepsilon^4 & \text{if } n=6.\end{cases}
\end{aligned}
$$

This completes the proof. □

The remaining cases are (M, g_0) is locally conformally flat or $3\le n\le 5$. Note that the above proof used the local geometry. So one might expect that the remaining cases can be considered via global geometry. The remaining cases, namely, when $3\le n\le 5$ or M is locally conformally flat was done by Schoen by using the positive mass theorem. Schoen solved this by using positive mass theorem and showed that $Y(M, g_0)<Y(\mathbb{S}^n, g_0)$. Then by Aubin-Trudinger, the Yamabe problem is solved. In next chapter, we will study the positive mass approach to the Yamabe problem.

Remark. There are another methods which can solve the Yamabe problem, i.e., there exists $g\in[g_0]$ such that R_g is constant. However, if $g=u^{\frac{4}{n-2}}g_0$, then $E(u)$ may not be minimized. Another method is the geometric flow which does not use positive mass theorem.

2.3 Boundary Yamabe Problem and Han-Li conjecture

This section is based on the guest lecture given by Professor Xuezhang Chen (Nanjing University).

2.3.1 Yamabe problem without boundary

Let (M^n, g_0) be a closed manifold. The Yamabe constant is defined by

$$
Y(M,[g_0])=\inf_{\tilde{g}\in[g_0]}\frac{\int_M R_{\tilde{g}}d\mu_{\tilde{g}}}{\left(\int_M d\mu_{\tilde{g}}\right)^{\frac{n-2}{n}}}.
$$

The conformal Laplacian is denoted by

$$L_{g_0} = -\frac{4(n-1)}{n-2}\triangle_{g_0} + R_{g_0}.$$

It is conformal invariant in the sense that for any $\varphi \in C^\infty(M)$,

$$L_{g_0}(u\varphi) = L_{u^{\frac{4}{n-2}}g_0}(\varphi) \quad \text{for some } 0 < u \in C^\infty(M).$$

In 1960, Yamabe [12] gives a proof. In 1968, Trudinger [11] found the error of the proof of Yamabe. Also, he proved that the Yamabe problem is true when $Y(M) \leq 0$ is true. In 1976, Aubin [2] proved the Yamabe problem if $n \geq 6$ and M is not locally conformally flat. In 1984, R. Schoen [10] proved the remaining cases: If $3 \leq n \leq 5$ or M is locally conformally flat. It is very natural to generalize to the compact manifold with boundary.

2.3.2 Boundary Yamabe problem

From now on, let (M^n, g_0) be a compact manifold with smooth boundary. We introduce some concepts corresponding to the (classical) Yamabe constant. First, we define

$$Y(M, \partial M, [g_0]) = \inf_{g\in[g_0]} \frac{\int_M R_g d\mu_g + 2(n-1)\int_{\partial M} h_g d\sigma_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}}.$$

Here h_g denotes a mean curvature on (M^n, g) which is defined by

$$h_g = \frac{1}{n-1}\mathrm{Tr}(\pi), \quad \pi_{ij} = \langle \nabla_i \nu_g, \partial_j \rangle, \quad 1 \leq i, j \leq n-1,$$

where ν_g is an outward unit normal on ∂M. Let $g = u^{\frac{4}{n-2}}g_0$. Then the corresponding the Euler-Lagrange equation is

$$\begin{cases} L_{g_0}u = c_1 u^{\frac{n+2}{n-2}} & \text{in } M, \\ B_{g_0}u = 0 & \text{on } \partial M. \end{cases}$$

Another Yamabe constant is defined by

$$Q(M, \partial M, [g_0]) = \inf_{g\in[g_0]} \frac{\int_M R_g d\mu_g + 2(n-1)\int_{\partial M} h_g d\sigma_g}{\left(\int_{\partial M} d\sigma_g\right)^{\frac{n-2}{n-1}}}.$$

Then the corresponding the Euler-Lagrange equation is

$$\begin{cases} L_{g_0}u = 0 & \text{in } M, \\ B_g u = \frac{n-2}{2}c_2 u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases}$$

We introduce a conformal invariant operator

$$B_{g_0} = \frac{\partial}{\partial \nu_g} + \frac{n-2}{2}h_{g_0}$$

in the sense that for any $\varphi \in C^\infty\left(\overline{M}\right)$,

$$B_{g_0}(u\varphi) = B_{u^{\frac{4}{n-2}}g_0}(\varphi) \quad \text{for any } 0 < u \in C^\infty\left(\overline{M}\right).$$

If the underlying metric g_0 is ambient, we drop $[g_0]$ in the notation.

Remark.

(i) $Y(M, \partial M) > -\infty$ and

(ii) $-\infty \leq Q(M, \partial M)$.

Problem. Let (M^n, g_0) be a smooth compact manifold with smooth boundary. Is there $g_0 \in [g_0]$ such that $R_g = c_2$ and $h_g = c_2$ for some $c_1, c_2 \in \mathbb{R}$?

We give a history. There are three types. (a) $c_1 = 0$, $c_2 \neq 0$ (b) $c_1 \neq 0$, $c_2 = 0$ (c) $c_1 \neq 0$, $c_2 \neq 0$.

(a) and (b) are started by Escobar [4] in 1992 and Escobar [5] in 1992. The strategy of Escobar was similar to the classical Yamabe problem. For (i), he made a criterion of the existence of the minimizer for $Q(M, \partial M)$. The model examples are $B_1(0) \subset \mathbb{R}^n$. Using the streographic projection $\pi : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$, we can pushforward $B_1(0)$ to $\mathbb{S}^n$. On the half space $\mathbb{R}^n_+ = \{x = (x', x^n) \in \mathbb{R}^n : x^n > 0\}$.

We introduce a bubble function

$$u_{\varepsilon, x_0'}(x) = \left(\frac{\varepsilon}{(\varepsilon + x^n)^2 + |x' - x_0|^2} \right)^{\frac{n-2}{2}}$$

for $\varepsilon > 0$, $x_0' \in \mathbb{R}^{n-1}$. Note that $u_{\varepsilon, x_0'}$ satisfies

$$\begin{cases} -\triangle u_{\varepsilon, x_0'} = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial u_{\varepsilon, x_0'}}{\partial x^n} = (n-2)\, u_{\varepsilon, x_0'}^{\frac{n}{n-2}} & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

For the case of unit ball, we can compute $Q\left(B^n, \partial B^n, |dx|^2\right)$ explicitly since $R_g = 0$ and $h_g = 1$. So

$$Q\left(B^n, \partial B^n, |dx|^2\right) = \frac{2(n-1)\,|\partial B^n|}{|\partial B^n|^{\frac{n-2}{n-1}}} = 2(n-1)\,|\partial B^n|^{\frac{1}{n-1}}.$$

We introduce one notion before to state the theorem.

Definition 2.22. Let π_{ij} denote the second fundamental form. We define $\mathring{\pi}_{ij} = \pi_{ij} - h_g g_{ij}$ the umbilicity tensor.

Remark. (i) If $\tilde{g} = \rho g$, $\rho > 0$, then $\mathring{\tilde{\pi}} = \rho\pi$(Exercise!)

(ii) For a Weyl tensor $W + A \owedge g = Rm$, we have $\widetilde{W}_{ijkl} = \rho W_{ijkl}$, where $A_{ij} = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R_g}{2(n-1)} g_{ij}\right)$ is the Schouten tensor.

Theorem 2.23 (Escobar [4]). *Let (M^n, g_0) be a compact smooth manifold with boundary, $n \geq 3$ with $Q(M, \partial M)$ finite. If (M^n, g_0) is not conformally equivalent to $\left(B^n, |dx|^2\right)$ and*

$$Q(M, \partial M) < Q(B^n, \partial B^n),$$

then there exists a minimizer of Q.

So we want to find a test function φ such that

$$Q(M, \partial M)$$

$$\leq \frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla g|^2_{g_0} + R_{g_0}\varphi^2\right) d\mu_{g_0} + 2(n-1)\int_{\partial M} h_g \varphi^2 d\sigma_{g_0}}{\left(\int_{\partial M} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}}$$
$$< Q(B^n, \partial B^n).$$

For (a), Escobar [4] proved the theorem is true if $n = 3$. It is also true if $n = 4, 5$ and ∂M is umbrlic. Also, if $n \geq 6$ and ∂M is umbilic with M is locally conformally flat.

In 2005, Marques [8] proved that the Yamabe problem is true for the following cases: If $n \geq 8$, M is umbilic and $\overline{W}(x_0) \neq 0$ for $x_0 \in \partial M$, where $\overline{W}$ denotes the Weyl tensor of submanifold. If $n \geq 9$, $W(x_0) \neq 0$ for $x_0 \in \partial M$, where ∂M is umbilic. In 2007, Marques [9] also proved the case when $n = 4, 5$ and ∂M has a non-umbilic point.

In 2010, Almaraz [1] proved the remaining case when $n = 6, 7, 8$, ∂M is umbilic $W(x_0) \neq 0$ for $x_0 \in \partial M$.

I will not mention the case (b). One can do this if one understand the case (a).

2.3.3 Han-Li conjecture

In 1999, Zhen-Chao Han and Yan Yan Li [6] proposed a conjecture:

Conjecture. If $Y(M, \partial M, [g_0]) > 0$, then there exists a conformal metric with constant scalar curvature 1 and any constant boundary mean curvature c.

The nonnegative case is easily solved by using subsolution method. In 1999, Han-Li [6] proved the conjecture when $n \geq 3$, M is locally conformally flat, and ∂M is umbilic. In 2000, Han-Li [7] proved the case ∂M has a non-umbilic point. There had been no improvement on this conjecture after these works. Chen, Ruan and Sun proved the following:

Theorem 2.24 (Chen-Ruan-Sun). *Let (M^n, g_0) be a smooth compact manifold with smooth boundary, $n \geq 3$. Assume that any of the following hypotheses holds:*

(i) $3 \leq n \leq 7$.

(ii) For $x_0 \in \partial M$, $d = \left[\frac{n-2}{2}\right]$, define

$$Z = \left\{ x_0 \in \partial M : 0 = \lim_{x \in M, x \to x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)| = \lim_{x \in \partial M, x \to x_0} d_{g_0}(x, x_0)^{1-d} |\mathring{\pi}_g(x)| \right\}.$$

Assume that $n \geq 8$, Z is non-empty and M is spin.

Then Han-Li conjecture is true. Moreover, if $n \geq 8$, ∂M is umbilic and the Weyl tensor is nonzero at $x_0 \in \partial M$, then Han-Li conjecture is true, in addition that $c \in [-c_0, \infty)$, where c_0 is a positive dimensional constant.

To prove this theorem, we recall original strategy of Han-Li: find a nontrivial Mountain Pass critcial point of a free functional:

$$I[u] := \int_M \left(\frac{4(n-1)}{n-2}|\nabla u|^2_{g_0} + R_{g_0}u^2\right) d\mu_{g_0} + 2(n-1)\int_{\partial M} h_{g_0} u^2 d\sigma_{g_0}$$
$$-4(n-1)(n-2)\int_M u_+^{\frac{2n}{n-2}} d\mu_{g_0} - 4c\int_{\partial M} u_+^{\frac{2(n-1)}{n-2}} d\sigma_{g_0},$$

where $u \in H^1(M, g_0)$, $u_+ = \max\{u, 0\}$, and $c \in \mathbb{R}$. The Euler-Lagrange equation for I is

$$-\frac{4(n-1)}{n-2}\triangle_{g_0} u + R_{g_0} u = 4n(n-1)\, u_+^{\frac{n+2}{n-2}} \quad \text{in } M \tag{2.20}$$

$$\frac{\partial u}{\partial \nu_{g_0}} + \frac{n-2}{2} h_{g_0} u = c u_+^{\frac{n}{n-2}} \quad \text{on } \partial M.$$

Let $g = u^{\frac{4}{n-2}} g_0$. Then $R_g = 4n(n-1)$ and $h_g = \frac{2c}{n-2} \in \mathbb{R}$.
We need a compactness lemma.

Lemma 2.25 (Compactness; Han-Li(weak-version)). *For any Palais-Samle sequence* $\{u_i\} \subset H^1(M, g_0)$ *satisfying*

$$I[u_i] \to L < s_c = I\left[W_\varepsilon; \mathbb{R}_+^n\right]$$

and

$$\|I'[u_i]\| \to 0 \quad \text{as } i \to \infty,$$

where W_ϵ *is the standard bubble. Then up to a subsequence, either* $\{u_i\}$ *goes to a nontrivial solution of* (2.20) *in* $H^1(M, g_0)$ *or* $\{u_i\} \to 0$ *in* $H^1(M, g_0)$.

So the goal is to find a test function φ such that

$$I_{mp} = \inf_{\overline{\gamma} \in \Gamma} \sup_{\psi \in \overline{\gamma}} I[\psi] \leq \max_{t \in [0,\infty)} I[t\varphi] < s_c,$$

where $\overline{\gamma}$ is a continuous path connecting 0 and φ and Γ is the space of such continuous paths.

For the case (i), Motivated by Simon Brendle [3], Chen and Sun constructed a test fuction

$$\varphi = \chi_\rho (W_\varepsilon + \phi) + (1 - \chi_\rho) G,$$

where χ_ρ is a cut-off function equal to 1 in $B_\rho^+$and 0 outside $B_{2\rho}^+$, G is a Green function for conformal Laplacian with conformally invariant boundary condition and ϕ is a correction term.

In dimension 7,

$$\varphi = \chi_\rho \left[\varepsilon^{\frac{n-2}{2}} \left(\mathcal{K}_2 (y^n - T_c\varepsilon)^2 + \mathcal{K}_1 \varepsilon (y^n - T_c\varepsilon) + \mathcal{K}_0 \right) \left(|y'|^2 + (y^n - T_c\varepsilon) \right)^{-\frac{n}{2}} \right],$$

where $y = (y', y^n) \in \mathbb{R}_+^n$.

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3 The Positive Mass Theorem and the resolution of the Yamabe problem

3.1 Asymptotically flat manifold and ADM mass

Definition 3.1 (Asymptotically flat)**.** A complete, noncompact, smooth Riemannian manifold (M, g) is called *asymptotically flat* (with one end) if there exists a compact set K in M such that $M \backslash K$ is diffeomorphic to $\mathbb{R}^n \backslash B_1(0)$, i.e., there exists a diffeomorphism $\varphi : M \backslash K \to \mathbb{R}^n \backslash B_1(0)$.

Let $x_1, \ldots, x_n$ be the pullback of Euclidean coordinates via φ, i.e.,

$$(x_1, \ldots, x_n) = \varphi^{-1}(\xi_1, \ldots, \xi_n),$$

where $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus B_1(0)$.

We require the metric g to satisfy

$$\begin{aligned} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g_{ij} &= \delta_{ij} + O\left(|x|^{-p}\right), \\ \partial g_{ij} &= O\left(|x|^{-p-1}\right), \\ \partial^2 g_{ij} &= O\left(|x|^{-p-2}\right) \end{aligned} \tag{3.1}$$

for some $p > \frac{n-2}{2}$ and

$$R_g = O\left(|x|^{-q}\right)$$

for some $q > n$. Here ∂_i denotes the covariant derivative.

We give some examples.

Example 3.2. $(\mathbb{R}^n, g_{\text{flat}})$ is asymptotically flat.

Example 3.3. $\left(\mathbb{R}^n \setminus \{0\}, \left(1 + \frac{n}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\text{flat}}\right)$, Schwarzchild spacetime, is asymptotically flat. Take $\varphi = \text{Id}$. We leave it to an exercise for conditions on the metric g. For the scalar curvature R_g,

$$-\frac{4(n-1)}{n-2} \triangle \left(1 + \frac{m}{2|x|^{n-2}}\right) + R_{g_{\text{flat}}} = R_g \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{n+2}{n-2}}.$$

Note that

$$\triangle \left(1 + \frac{m}{2|x|^{n-2}}\right) = 0.$$

Hence $R_g = 0$. Observe also that Schwarzchild spacetime is rotational symmetric.

Remark. (i) For simplicity, we write

$$g_{ij} = \delta_{ij} + O_2\left(|x|^{-p}\right)$$

for (3.1).

(ii) Note that the condition "$g_{ij} = \delta_{ij} + O\left(|x|^{-p}\right)$" does not imply the other conditions. For example, take $g_{ij} = \left(1 + \sin(x_1^n) / |x_1|^{n/2}\right) \delta_{ij}$. Then $g_{ij} = \delta_{ij} + O\left(|x|^{-n/2}\right)$. But

$$\partial_1 g_{ij} = \left(\frac{n x_1^{n-1} \cos(x_1^n)}{|x_1|^{n/2}} + \cdots\right) \delta_{ij}$$

$$= O\left(|x_1|^{n/2-1}\right)$$

since the sine function osculates too much.

Some other papers or books use different asymptotic orders. They require

$$\begin{aligned} g_{ij} &= \delta_{ij} + O_2\left(|x|^{-p}\right) \\ R_g &= O\left(|x|^{-q}\right) \end{aligned}$$

for some $p > p'$ and $q > q'$. Here p' and q' may not be $(n-2)/2$ or n, respectively. The reason is historical because they didn't know the exact order in the definition. Our definition is natural in the view of the following definition.

Definition 3.4. The *ADM mass* (introduced by R. Arnowitt, S. Deser, and C. W. Misner in 1959) for an asymptotically flat manifold (M, g) is defined by

$$m_{\mathrm{ADM}}(M, g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\sigma\to\infty} \int_{|x|=\sigma} \left(\partial_i g_{ij} - \partial_j g\right) \nu^j \, dS,$$

where ν is the outward unit normal of $\{|x| = \sigma\}$ and ω_{n-1} is the volume of $(n-1)$-dimensional unit sphere.

Example 3.5. Take $(M, g) = (\mathbb{R}^n, g_{\text{flat}})$. As we know, $\partial_i g_{ij} = \partial_j g_{ii} = 0$. So $m_{\mathrm{ADM}}(\mathbb{R}^n, g_{\text{flat}}) = 0$.

Example 3.6. $(M, g) = \left(\mathbb{R}^n \setminus \{0\}, \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\text{flat}}\right)$, Schwarzchild spacetime

$$\begin{aligned} g_{ij} &= \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij} \\ \partial_i g_{ij} &= \frac{\partial}{\partial x_i}\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij} = -2m\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}-1} |x|^{-n} x_i \delta_{ij} \\ \partial_j g_{ii} &= -2m\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}-1} |x|^{-n} x_j. \end{aligned}$$

Note that the outward unit normal is given by $\nu = \frac{x}{|x|}$. This implies

$$\begin{aligned} \left(\partial_i g_{ij} - \partial_j g\right)\nu^j &= -2m\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}-1} |x|^{-n} \left[x_i\delta_{ij} - x_j\right] \frac{x_j}{|x|} \\ &= -2m\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}-1} |x|^{-n} \left(\frac{|x|^2}{|x|} - n\frac{|x|^2}{|x|}\right) \\ &= -2m(n-1)\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}-1} |x|^{1-n}. \end{aligned}$$

So

$$\begin{aligned} m_{\mathrm{ADM}}(\text{Schwardchild}) &= \frac{m}{\omega_{n-1}} \lim_{\sigma\to\infty} \int_{\mathbb{S}^{n-1}} \left(1 + \frac{m}{2|\sigma|^{n-2}}\right)^{\frac{4}{n-2}-1} |\sigma|^{1-n} |\sigma|^{n-1} \, dV_{g_{\mathbb{S}^{n-1}}} \\ &= m \lim_{\sigma\to\infty}\left(1 + \frac{m}{2\sigma^{n-2}}\right)^{\frac{4}{n-2}-1} = m. \end{aligned}$$

Our asympotitc order on a metric and the scalar curvature gurantees the well-definedness of ADM mass. To see this, Stokes' theorem gives

$$\begin{aligned}&\int_{|x|=\sigma_2} (\partial_i g_{ij} - \partial_j g_{ii}) \nu_j ds - \int_{|x|=\sigma_1} (\partial_i g_{ij} - \partial_j g_{ii}) \nu_j ds \\ &= \int_{\sigma_1<|x|<\sigma_2} (\partial_j \partial_i g_{ij} - \partial_j \partial_j g_{ii}) dV.\end{aligned}$$

If $\sum_{i,j} (\partial_j \partial_i g_{ij} - \partial_j \partial_j g_{ii})$ is an integrable function, then the above boundary integral on balls of radius σ is a Cauchy sequence and hence has a limit. So we have to prove that $\sum_{i,j} (\partial_j \partial_i g_{ij} - \partial_j \partial_j g_{ii})$ is integrable.

Recall, in local coordinates, we have

$$R_g = \sum_{i,j=1}^{n} (\partial_i \partial_j g_{ij} - \partial_i \partial_j g_{ii}) + O\left((g-\delta)\, \partial^2 g\right) + O\left((\partial g)^2\right).$$

Since (M, g) is asymptotic flat,

$$O\left(|x|^{-q}\right) = \sum_{i,j=1}^{n} (\partial_i \partial_j g_{ij} - \partial_i \partial_j g_{ii}) + O\left(|x|^{-p} |x|^{-p-2}\right) + O\left(\left(|x|^{-p-1}\right)^2\right)$$

for some $q > n$ and $p > \frac{n-2}{2}$. So

$$\begin{aligned}\sum_{i,j=1}^{n} (\partial_i \partial_j g_{ij} - \partial_i \partial_j g_{ii}) &= O\left(|x|^{-q}\right) + O\left(|x|^{-p} |x|^{-p-2}\right) + O\left(\left(|x|^{-p-1}\right)^2\right) \\ &= O\left(|x|^{-r}\right)\end{aligned}$$

for some $r > n$, which shows the claim.

3.2 The Positive Mass Theorem

In this section, we prove the positive mass theorem, which plays a central role in the resolution of Yamabe problem.

Theorem 3.7. *If (M, g) is an asymptotically flat manifold with $R_g \geq 0$ with $\dim M = n$, then $m_{\mathrm{ADM}}(M, g) \geq 0$. Moreover, $m_{\mathrm{ADM}}(M, g) = 0$ if and only if (M, g) and $(\mathbb{R}^n, \delta)$ are isometric, where δ denote the flat metric in $\mathbb{R}^n$.*

One surprising things of this theorem is that the zero mass of ADM determines the geometry. Historically, this theorem is proved by Scheon and Yau in 1979 and 1980 [11, 12] for the case $3 \leq \dim M = n \leq 7$ by using minimal hypersurface. Few year laters, in 1984, Edward Witten [15] proved the PMT when (M, g) is spin (but for all n) by using spinor. Very recently, in 2017, Schoen-Yau [9] claim that the PMT is true in full generality. In this course, we prove the case when $3 \leq n \leq 7$. The proof is very elegant and beautiful.

Before to prove the PMT, we give some interesting consequences.

- In $(\mathbb{R}^n, \delta)$, let K be a compact subset of $\mathbb{R}^n$. Perturb δ in a compact set K to g, while keeping it the same in $\mathbb{R}^n \setminus K$. We claim that we cannot perturb δ to g in a compact set K such that $R_g \geq 0$. Indeed, if it is possible, then $m_{\mathrm{ADM}}(\mathbb{R}^n, g) = 0$ since $g = \delta$ in $\mathbb{R}^n \setminus K$. By equality part of PMT, $g = \delta$ in $\mathbb{R}^n$. Before the positive mass theorem, many people did not expect this result.

- Consider $(\mathbb{R}^n, g)$ with $R_g \geq 0$. If $g = \delta$ in $\mathbb{R}^n \setminus K$, then $g = \delta$ in $\mathbb{R}^n$. Hitchin mentioned that this cannot be true. But it actually true by positive mass theorem.

One motivation of the positive mass theorem is the following: in general relativity, R_g is related to local density. The m_{ADM} is related to the global mass. So one might ask whether the global mass is nonnegative if the local density is nonnegative.

Remark. This is sometimes called a Riemannian positive mass theorem. There is a more general version of PMT, positive energy theorem, for pseudo-Riemannian manifold. In 2016, Eichmair-Huang-Lee-Schoen [2] proved the spacetime positive mass theorem in dimensions less than 8.

3.2.1 Steps for proving the positive mass theorem

There are five steps to prove this theorem.

Step 1. It suffices to consider the case when $R_g \equiv 0$. Indeed, since $R_g \geq 0$, the conformal Laplacian given by

$$L = -\frac{4(n-1)}{n-2}\triangle_g + R_g$$

is positive definite. We can find a positive solution u such that

$$-\frac{4(n-1)}{n-2}\triangle_g u + R_g u = 0 \quad \text{in } M. \tag{3.2}$$

We asume the following facts:

Fact 1. If we normalize the asymptotic condition to be $u \to 1$ as $|x| \to \infty$, then u is uniquely determined.

The fact is essentially maximum principle.

Fact 2. We have the asymptotic expansion of u at ∞:

$$u(x) = 1 + \frac{A}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-1}}\right). \tag{3.3}$$

Assuming these facts, we claim that $A \leq 0$. Integrating (3.2) over B_σ gives

$$0 = \int_{B_\sigma}\left(-\frac{4(n-1)}{n-2}\triangle_g u + R_g u\right) dV_g.$$

Integration by parts gives

$$\int_{\partial B_\sigma}\frac{4(n-1)}{n-2}\frac{\partial u}{\partial \nu_g} dS = \int_{B_\sigma} R_g u dV_g \geq 0 \tag{3.4}$$

since $R_g \geq 0$ and $u > 0$. On the other hand, by (3.3),

$$\begin{aligned}\frac{\partial u}{\partial x_i} &= A\left(-\frac{n-2}{2}\right)\left(|x|^2\right)^{-\frac{n-2}{2}-1}(2x_i) + O\left(-\frac{n-1}{2}\left(|x|^2\right)^{-\frac{n-1}{2}-1}(2x_i)\right)\\ &= -(n-2)A|x|^{-n}x_i + O\left(|x|^{-n}\right).\end{aligned}$$

Since (M, g) is asymptotically flat,

$$\nu_g \sim \frac{x}{|x|}.$$

So

$$\frac{\partial u}{\partial \nu_g} = \nabla u \cdot \nu_g \sim -(n-2)\, A\, |x|^{-n}\, |x| + O\left(|x|^{-n}\right).$$

Hence

$$\begin{aligned}\int_{\partial B_\sigma} \frac{\partial u}{\partial \nu_g} dS_g &\sim \int_{|x|=\sigma} \left(-(n-2)\, A\, |x|^{1-n} + O\left(|x|^{-n}\right)\right) dS_g \\ &= \omega_{n-1}\left(-(n-2)\, A\sigma^{1-n} + O\left(\sigma^{-n}\right)\right)\sigma^{n-1} \\ &= -(n-2)\,\omega_{n-1} A + O\left(\sigma^{-1}\right).\end{aligned}$$

Put this into (3.4),

$$-\omega_{n-1}(n-2)\, A + O\left(\sigma^{-1}\right) = \int_{\partial B_\sigma} \frac{\partial u}{\partial \nu_g} dS_g \geq 0.$$

Lettting $\sigma \to \infty$, we get $A \leq 0$, which proves the claim.

Now we define the metric by

$$\tilde{g} = u^{\frac{4}{n-2}} g.$$

Then $R_{\tilde{g}} = 0$ by (3.2). By using (3.3), one can check that $\tilde{g}$ is still asymptotically flat (Exercise!) and

$$m_{\mathrm{ADM}}(\tilde{g}) = m_{\mathrm{ADM}}(g) + 2A. \tag{3.5}$$

Since $A \leq 0$ by claim, $m_{\mathrm{ADM}}(\tilde{g}) \leq m_{\mathrm{ADM}}(g)$. In particular, if we can prove the positive mass theorem for $(M, \tilde{g})$, the positive mass theorem for (M, g) follows. Let us prove (3.5). Note

$$\begin{aligned}\partial_i \tilde{g}_{ij} &= \partial_i \left(u^{\frac{4}{n-2}} g_{ij}\right) \\ &= \frac{4}{n-2} u^{\frac{4}{n-2}-1} \partial_i u g_{ij} + u^{\frac{4}{n-2}} \partial_i g_{ij}\end{aligned} \tag{3.6}$$

and

$$\partial_j \tilde{g}_{ii} = \frac{4}{n-2} u^{\frac{4}{n-2}-1} \partial_j u g_{ii} + u^{\frac{4}{n-2}} \partial_j g_{ii}. \tag{3.7}$$

By (3.3)

$$\partial_i u = -(n-2)\, A\, |x|^{-n}\, x_i + O\left(|x|^{-n}\right).$$

Set

$$\tilde{\nu} = u^{-\frac{2}{n-2}} \nu.$$

Then we have

$$\tilde{g}(\tilde{\nu}, \tilde{\nu}) = u^{\frac{4}{n-2}} g\left(u^{-\frac{2}{n-2}}\nu, u^{-\frac{2}{n-2}}\nu\right) = 1.$$

and

$$dS_{\tilde{g}} = u^{\frac{2}{n-2}(n-1)} dS_g.$$

So by definition of ADM mass, (3.6), and (3.7), we have

$$\begin{aligned}& m_{\mathrm{ADM}}(\tilde{g}) \\ &= \frac{1}{2(n-1)\,\omega_{n-1}} \lim_{\sigma\to\infty} \int_{|x|=\sigma} \left(\partial_i \tilde{g}_{ij} - \partial_j \tilde{g}_{ii}\right) \tilde{\nu}\, dS_{\tilde{g}} \\ &= \frac{1}{2(n-1)\,\omega_{n-1}} \lim_{\sigma\to\infty} \int_{|x|=\sigma} \left[\frac{4}{n-2} u^{\frac{4}{n-2}-1}\left(\partial_i u g_{ij} - \partial_j u g_{ii}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+u^{\frac{4}{n-2}}\left(\partial_i g_{ij}-\partial_j g_{ii}\right)\Big] u^{-\frac{2}{n-2}}\nu^j u^{-\frac{2(n-1)}{n-2}}dS_g \\
&= m_{\mathrm{ADM}}(g) \\
&+\frac{1}{2(n-1)\omega_{n-1}}\lim_{\sigma\to\infty}\int_{|x|=\sigma}\left[\frac{4}{n-2}u^{\frac{4}{n-2}-1}\left(\partial_i u g_{ij}-\partial_j u g_{ii}\right)\right]u^{-\frac{2}{n-2}}\nu^j u^{-\frac{2(n-1)}{n-2}}dS_g
\end{aligned}
$$

Now under some reduction and taking change of variables into a polar coordinate, we have

$$
\begin{aligned}
&\frac{1}{2(n-1)\omega_{n-1}}\lim_{\sigma\to\infty}\int_{|x|=\sigma}\left[\frac{4}{n-2}u^{\frac{4}{n-2}-1}\left(\partial_i u g_{ij}-\partial_j u g_{ii}\right)\right]u^{-\frac{2}{n-2}}\nu^j u^{-\frac{2(n-1)}{n-2}}dS_g \\
&=\frac{1}{2(n-1)\omega_{n-1}}\lim_{\sigma\to\infty}\int_{|x|=\sigma}\left[\left(-(n-2)A|x|^{-n}x_i+O\left(|x|^{-n}\right)\right)\left(\delta_{ij}+O\left(|x|^{-p}\right)\right)\right. \\
&\left.-(n-2)A|x|^{-n}x_j+O\left(|x|^{-n}\right)\left(\delta_{ii}+O\left(|x|^{-p}\right)\right)\right]\frac{x_j}{|x|}dS_g \\
&=\frac{4}{2(n-1)(n-2)}\lim_{\sigma\to\infty}\left[\left(-(n-2)A|x|^{-n}x_i+O\left(|x|^{-n}\right)\right)\left(\delta_{ij}+O\left(|x|^{-p}\right)\right)\right. \\
&\quad\left.\left.-\left(-(n-2)A|x|^{-n}x_j+O\left(|x|^{-n}\right)\left(\delta_{ii}+O\left(|x|^{-p}\right)\right)\right)\frac{x_j}{|x|}|x|^{n-1}\right]\right|_{|x|=\sigma} \\
&=\frac{4}{2(n-1)(n-2)}\lim_{\sigma\to\infty}\left.-(n-2)A|x|^{-n}\frac{|x|^2}{|x|}|x|^{n-1}+n(n-2)A|x|^{-n}\frac{|x|^2}{|x|}|x|^{n-1}\right|_{|x|=\sigma} \\
&=2A
\end{aligned}
$$

for some $p>\frac{n-2}{2}$. This completes Step 1.

We sketch the proof of Step 2 and Step 3.

Step 2. we can approximate g by another $\overline{g}$ which is *harmonically flat* near infinity and having the ADM mass closed to the ADM mass of g. More precisely, given $\varepsilon>0$, there exists $\overline{g}$ such that

(i) $(M,\overline{g})$ is asymptotically flat,

(ii) $(M,\overline{g})$ is scalar flat, i.e., $R_{\overline{g}}\equiv 0$,

(iii) $(M,\overline{g})$ is *conformally flat* near infinity, i.e.,

$$\overline{g}=u^{\frac{4}{n-2}}\delta.$$

(iv) $|m_{\mathrm{ADM}}(g)-m_{\mathrm{ADM}}(\overline{g})|<\varepsilon$.

The idea of the proof is the following. Replace the metric by δ near infinity and restore the condition $R\equiv 0$ by conformal method. More precisely, for any large $\sigma>0$, we define a cut-off function ψ_σ by

$$\psi_\sigma(|x|)=\begin{cases}1 & \text{if } |x|<\sigma\\ 0 & \text{if } |x|>2\sigma\end{cases}$$

with $\psi'_\sigma\le 0$,$|\psi'_\sigma|\le C/\sigma$ and $|\psi''_\sigma|\le C/\sigma^2$. Define $g^{(\sigma)}$ by

$$g^{(\sigma)}=\psi_\sigma(\delta+g)+(1-\psi_\sigma)\delta.$$

Exercise 3.8. Check that $g^{(\sigma)}$ is harmonically flat. Also, show that $m_{\mathrm{ADM}}(g)$ is sufficiently close to $m_{\mathrm{ADM}}(\overline{g})$.

Why do we call the harmonic flat? The justification of the term is the following. By (ii) and (iii), Yamabe equation, we have

$$-\frac{4(n-1)}{n-2}\triangle_\delta u + R_\delta = R_{\overline{g}} u^{\frac{n+2}{n-2}}$$

near infinity. So u is harmonic near at infinity. This completes Step 2.

Step 3. Aiming for contradiction, given $\overline{g}$ in Step 2 with $m_{\mathrm{ADM}}(\overline{g}) < 0$, we can find $\tilde{g}$ such that

(i) $(M, \tilde{g})$ has $R_{\tilde{g}} \geq 0$, and

(ii) $\tilde{g}$ is Euclidean outside a compact set.

The idea of the proof is the following.

Fact. We have the asymptotics:

$$u(x) = 1 + \frac{m}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-1}}\right).$$

So

$$m_{\mathrm{ADM}}(\overline{g}) = m_{\mathrm{ADM}}\left(u^{\frac{4}{n-2}}\delta\right) = m.$$

By assumption, we have $m < 0$. By the asymptotics for u and for $\sigma \gg 0$, we have

$$1 + \frac{m}{\sigma^{n-2}} \sim \max_{S_\sigma} u < \min_{S_{2\sigma}} u \sim 1 + \frac{m}{(2\sigma)^{n-2}}$$

where $S_\sigma = \{|x| = \sigma\}$. Fix such a σ and some constant $\alpha \in (\max_{S_\sigma} u, \min_{2\sigma} u)$. Let $v = \min\{u, \alpha\}$. Then v is superharmonic in C^0-sense (exercise). In fact, we can find smooth $\tilde{v} > 0$ which is still superharmonic and

$$\begin{cases} \tilde{v} = u & \text{in } B_\sigma \\ \tilde{v} = \alpha & \text{in } M \setminus B_{2\sigma}. \end{cases}$$

Therefore, if we define

$$\tilde{g} = \begin{cases} g & \text{in } B_\sigma, \\ \tilde{v}^{\frac{4}{n-2}}\delta & \text{in } M \setminus B_{2\sigma}. \end{cases}$$

Then $\tilde{g}$ satisfies (i) and (ii). This completes Step 3.

Remark. $\tilde{g}$ is non-flat since g is not flat in B_σ.

Step 4. If there exists an asymptotically flat (M, g) with $m_{\mathrm{ADM}}(g) < 0$, then there exists an asymptotically flat $(M, \tilde{g})$ which is non-flat Euclidean outside a compact set and has $R_{\tilde{g}} \geq 0$.

Now the geometric idea comes in. This idea comes from Lohkamp. Take a big cube in the flat region which encloses the non-flat part M_0 of M. Then identifying the boundaries of the cube, we get a compact manifold M', which is a connected sum of a flat torus $\mathbb{T}^n$ with M_0 which admits a non-flat metric g with nonnegative scalar curvature.

The following lemma is classical.

Lemma 3.9. *Let (M, g) be a compact manifold with $R_g \geq 0$ which is not Ricci-flat, i.e.,* $\mathrm{Ric}_g \neq 0$. *Then g can be perturbed to a metric $\overline{g}$ with $R_{\overline{g}} > 0$.*

The modern approach to prove Lemma 3.9 is to use Ricci flow. We will prove this in later.

Lemma 3.10. *The (M', g) we constructed before is not Ricci-flat.*

From Lemma 3.9 and Lemma 3.10, there exists $\overline{g}$ in $M' = \mathbb{T}^n \# M_0$ such that $R_{\overline{g}} > 0$.
There is a conjecture suggested by Lohkamp.

Conjecture (Lohkamp)**.** For any compact n-dimensional manifold M, the connected sum $\mathbb{T}^n \# M$ does not have a metric of positive scalar curvature.

Remark. Lohkamp's conjecture implies that the positive mass theorem is true. The conjecture is not artifical. Note that

$$\mathbb{T}^n \# M = \mathbb{T}^n$$

which has not metric of positive scalar curvature. This was proved by Schoen and Yau.

Now we conclude Step 4 by proving Lemma 3.9 and Lemma 3.10.

Proof of Lemma 3.9. We use the Ricci flow

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2 \operatorname{Ric}_{g(t)} \quad \text{for } t \geq 0 \\ g(t)\,|_{t=0} &= g. \end{aligned} \tag{3.8}$$

This was first introduced by Richard Hamilton in 1982 [6]. Hamilton proved the short-time existence of the Ricci flow, i.e., there exists smooth $g(t)$ on $M \times [0, \varepsilon)$ which satisfies (3.8) for some $\varepsilon > 0$. Note that this equation is of not parabolic type. Along the Ricci flow (3.8),

$$\frac{\partial}{\partial t} R_{g(t)} = \triangle_{g(t)} R_{g(t)} + 2 \left| \operatorname{Ric}_{g(t)} \right|^2_{g(t)}.$$

Based on these facts, we are ready to prove Lemma 3.9. Given (M, g) with $R_g \geq 0$ and $|\operatorname{Ric}_g|^2_g > 0$, we can start the flow with $g(t)\,|_{t=0} = g$. By the maximum principle, $R_{g(t)} > 0$ or $R_{g(t)} \equiv 0$ and $\operatorname{Ric}_{g(t)} \not\equiv 0$. ☐

Proof of Lemma 3.10. First, we look at the degree 1 map from M to $\mathbb{T}^n$ by contracting M_0 to a point . Denote this by $\pi : M' \to \mathbb{T}^n$. Without loss of generality, we assume that $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, then we have 1-forms on $\mathbb{T}^n$ given by

$$\alpha_i = dx_i, \quad 1 \leq i \leq n.$$

Note that α_i's are all closed and not exact (Exercise!). So $0 \neq [\alpha_i] \in H^1_{\mathrm{DR}}(\mathbb{T}^n)$, where $H^k_{\mathrm{DR}}(M)$ denotes kth de Rham cohomology of M. In fact, $\{[\alpha_i]\}_{i=1}^n$ forms a basis for $H^1_{\mathrm{DR}}(\mathbb{T}^n)$. Also,

$$\int_{\mathbb{T}^n} \alpha_1 \wedge \cdots \wedge \alpha_n = 1.$$

Using pullback, we obtain $\theta_i = \pi^* \alpha_i$, $1 \leq i \leq n$. Then θ_i is closed 1-form in M'. We can compute

$$\begin{aligned} \int_{M'} \theta_1 \wedge \cdots \wedge \theta_n &= \int_{M'} \pi^* \alpha_1 \wedge \cdots \wedge \pi^* \alpha_n \\ &= \int_{M'} \pi^* (\alpha_1 \wedge \cdots \wedge \alpha_n) \\ &= \deg(\pi) \int_{\mathbb{T}^n} (\alpha_1 \wedge \cdots \wedge \alpha_n) = 1 \cdot 1 = 1. \end{aligned}$$

So θ_i is not exact for $1 \leq i \leq n$ by Stokes' theorem. This implies $0 \neq [\theta_i] \in H^1_{\mathrm{DR}}(M')$ for $1 \leq i \leq n$. By the Hodge Theorem, we can find a harmonic 1-form θ_i^H in M' such that $\left[\theta_i^H\right] = [\theta_i]$, i.e., $\triangle \theta_i^H = 0$. By Bochner's formula, we have

$$\begin{aligned} \frac{1}{2} \triangle \left|\theta_i^H\right| &= -\left|\nabla \theta_i^H\right|^2 + \theta_i \triangle \theta_i^H + \mathrm{Ric}\left(\nabla \theta_i, \nabla \theta_i\right) \\ &= -\left|\nabla \theta_i^H\right|^2 \end{aligned}$$

if we assume M' is Ricci-flat. Integrating this over M', Stokes' theorem gives

$$0 = \int_{M'} \left|\nabla \theta_i^H\right|^2$$

and so $\nabla \theta_i^H \equiv 0$ in M', i.e., $\left\{\theta_1^H, \ldots, \theta_n^H\right\}$ are parallel 1-form in M'. Since $[\theta_i] = \left[\theta_i^H\right]$ for all $1 \leq i \leq n$, $\theta_i = \theta_i^H + d\beta_j$ for some 0-form β_j in M'. So

$$\theta_1 \wedge \cdots \wedge \theta_n = \theta_1^H \wedge \cdots \wedge \theta_n^H + d\beta$$

for some $(n-1)$-form β in M'. So

$$1 = \int_{M'} \theta_1 \wedge \cdots \wedge \theta_n = \int_{M'} \theta_1^H \wedge \cdots \wedge \theta_n^H.$$

Hence $\left\{\theta_1^H, \ldots, \theta_n^H\right\}$ are parallel 1-form in M', which forms a basis of cotangent bundle $(TM')^*$. This implies that M' is flat. But this is a contradiction since $M' = \mathbb{T}^n \# M_0$ is not flat. This completes the proof. □

Step 5. We construct a minimal hypersurface Σ in M'. To do this, we will apply the geometric measure theory. Recall, in the proof of Lemma 3.10, we have $\alpha_1, \ldots, \alpha_n$ closed 1-form in $\mathbb{T}^n$ and not exact. Consider

$$0 \neq [\alpha_1 \wedge \cdots \wedge \alpha_{n-1}] \in H^{n-1}_{\mathrm{DR}}(\mathbb{T}^n)$$

is dual to the sub-torus

$$\mathbb{T}^{n-1} = \{x_n = 0\}$$

in the sense that

$$\int_{\mathbb{T}^{n-1}} \alpha_1 \wedge \cdots \wedge \alpha_{n-1} = 1.$$

This implies that $[\alpha_1 \wedge \cdots \wedge \alpha_{n-1}]$ is an integral cohomology class of $\mathbb{T}^n$, i.e.,

$$[\alpha_1 \wedge \cdots \wedge \alpha_{n-1}] \in H^{n-1}(\mathbb{T}^n, \mathbb{Z}).$$

The pullback gives

$$\pi^* [\alpha_1 \wedge \cdots \wedge \alpha_{n-1}] = [\pi^* (\alpha_1 \wedge \cdots \wedge \alpha_{n-1})] = [\theta_1 \wedge \cdots \wedge \theta_{n-1}] \in H^{n-1}(M', \mathbb{Z}).$$

Since $\pi : M' \to \mathbb{T}^n$, $\pi^* : H^{n-1}(\mathbb{T}^n, \mathbb{Z}) \to H^{n-1}(M', \mathbb{Z})$. So $[\theta_1 \wedge \cdots \wedge \theta_{n-1}] \in H^{n-1}(M', \mathbb{Z})$ is dual to a homology class $\sigma_{n-1} \in H_{n-1}(M', \mathbb{Z})$. By geometric measure theory(see Federer-Fleming [4] or Federer [3]), we can minimize the volume among all submanifolds in the same integral homology class, i.e., there exists $\Sigma^{n-1} \in \sigma_{n-1}$ such that

$$|\Sigma| = \min\{|\Sigma_0| : \Sigma_0 \in \sigma_{n-1}\}$$

and

$$\int_{\Sigma} \theta_1 \wedge \cdots \wedge \theta_{n-1} = 1.$$

In general, Σ^{n-1} is not smooth. In fact, we know that Σ^{n-1} is smooth away from a set of comension 8. If $3 \leq n \leq 7$, then Σ^{n-1} is smooth.

Consider the variation of Σ^{n-1} in M'. Fix an orthonormal frame $\{e_1, \ldots, e_n\}$ in M' such that $\{e_1, \ldots, e_{n-1}\}$ is a frame tangent to Σ^{n-1} and e_n is an unit normal vector field on Σ^{n-1}. We will vary the surface Σ^{n-1} in the direction e_n. For any $\varphi \in C^\infty\left(\Sigma^{n-1}\right)$, we can consturct a familiy of hypersurfaces

$$\Sigma_t^{n-1} = \exp\left(t\varphi e_n\right)\Sigma^{n-1}.$$

A direct computation shows that

$$\frac{d}{dt}\left|\Sigma_t^{n-1}\right|\Big|_{t=0} = 0 \ \Rightarrow\ H \equiv 0$$

and

$$\frac{d^2}{dt^2}\left|\Sigma_t^{n-1}\right|\Big|_{t=0} \geq 0 \quad \text{if } \Sigma^{n-1} \text{ is a minimizer,} \tag{3.9}$$

where H is the mean curvature of Σ^{n-1}. Note that

$$\begin{aligned}
&\frac{d^2}{dt^2}\left|\Sigma_t^{n-1}\right| \\
&= \int_{\Sigma^{n-1}}\left(|\nabla\varphi|^2 - \operatorname{Ric}^M\left(e_n, e_n\right)\varphi^2 - |A|^2\varphi^2\right)d\mu,
\end{aligned}$$

where A is the 2nd fundamental form.

Definition 3.11. A minimal hypersurface Σ^{n-1} is *stable* if (3.9) is satisfied for all $\varphi \in C^\infty\left(\Sigma^{n-1}\right)$.

In particular, a minimizer is stable. We can rewrite (3.9) as follows:

$$\begin{aligned}
R^{M'} &= \sum_{a,b=1}^{n} R_{abab}^{M'} \\
&= \sum_{i,j=1}^{n-1} R_{ijij}^{M'} + 2\operatorname{Ric}^{M'}\left(e_n, e_n\right) \\
&= \sum_{i,j=1}^{n-1}\left(R_{ijij}^{\Sigma^{n-1}} - A_{ii}A_{jj} + A_{ij}^2\right) + 2\operatorname{Ric}^{M'}\left(e_n, e_n\right) \\
&= R^{\Sigma^{n-1}} - H^2 + |A|^2 + 2\operatorname{Ric}^{M'}\left(e_n, e_n\right).
\end{aligned}$$

Here we used the Gauss equation. Put this into (3.9), we get

$$\int_{\Sigma^{n-1}}\left(|\nabla\varphi|^2 - \frac{1}{2}\left(R^{M'} - R^{\Sigma^{n-1}} + |A|^2\right)\varphi^2\right)d\mu \geq 0$$

So

$$\int_{\Sigma}\left(|\nabla\varphi|^2 + \frac{1}{2}R^{\Sigma^{n-1}}\varphi^2\right)d\mu \geq \int_{\Sigma}\left(\frac{1}{2}R^{M'}\varphi^2 + \frac{1}{2}|A|^2\varphi^2\right)d\mu > 0.$$

for any $0 \not\equiv \varphi \in C^\infty\left(\Sigma^{n-1}\right)$. Thus, the first eigenvalue of $-\triangle_{\Sigma^{n-1}} + \frac{1}{2}R^{\Sigma^{n-1}}$ is positive.

Consider the conformal Laplacian of Σ^{n-1}, i.e.,

$$-\frac{4\left(n-2\right)}{n-3}\triangle_{\Sigma} + R^{\Sigma^{n-1}}.$$

Since $\frac{2(n-2)}{n-3} \geq 1$, we have

$$\begin{aligned}\lambda_1\left(-\frac{4(n-1)}{n-2}\triangle_{\Sigma^{n-1}}+R^{\Sigma^{n-1}}\right) &= \lambda_1\left(2\left(-\frac{2(n-2)}{n-3}\right)\triangle_{\Sigma}+\frac{1}{2}R^{\Sigma^{n-1}}\right)\\ &\geq \lambda_1\left(2\left(-\triangle_{\Sigma}+\frac{1}{2}R^{\Sigma^{n-1}}\right)\right)\end{aligned}$$

In particular, let u be the 1st eigenvalue of

$$-\frac{4(n-2)}{n-3}\triangle_{\Sigma}+R^{\Sigma^{n-1}}.$$

Then $0<u\in C^{\infty}\left(\Sigma^{n-1}\right)$ and $\tilde{g}=u^{\frac{4}{n-3}}g_{\Sigma^{n-1}}$ has positive scalar curvature.

So we can play the same game for Σ^{n-1} to get an $(n-2)$-submanifold Σ^{n-2} which has the same conditions. Applying the argument repeatedly, we get a 2-dimensional manifold $\left(\Sigma^2, g\right)$ with $R_g>0$ and $\int_{\Sigma^2}\theta^1\wedge\theta^2=1$.

Since $R_g>0$, the Gaussian curvature K_g is positive. Hence, by the Gauss-Bonnet theorem, we have

$$2\pi\chi\left(\Sigma^2\right)=\int_{\Sigma^2}K_g>0,$$

where $\chi(M)$ denotes the Euler characteristic of a manifold M. Hence, Σ^2 is diffeomorphic to $\mathbb{S}^2$. Since θ^1, θ^2 are closed 1-form in $\mathbb{S}^2$ and $H^1_{\mathrm{DR}}\left(\mathbb{S}^2,\mathbb{R}\right)=\{0\}$, θ^1 and θ^2 are exact. So $\theta^1=df^1$ and $\theta^2=df^2$ for some $f^1, f^2\in C^{\infty}\left(\mathbb{S}^2\right)$. Hence,

$$\int_{\mathbb{S}^2}\theta^1\wedge\theta^2=0$$

by Stokes' theorem, which is a contradiction. This proves the inequality of PMT, namely, if (M,g) is asymptotically flat with $R_g\geq 0$ and $3\leq n\leq 7$.

For the equality part, we have the following theorem.

Theorem 3.12. *If (M,g) has $m_{\mathrm{ADM}}(g)=0$, then it is Ricci-flat.*

We can prove this theorem by using Ricci flow. We omit the proof. Accepting this theorem, for $\dim M=3$, then the sectional curvature is zero, i.e., g is flat. Hence, (M,g) is isometric to $\left(\mathbb{R}^3,\delta\right)$. This completes the proof of the positive mass theorem.

3.3 The resolution of Yamabe problem

So now we come back to the Yamabe problem. For $Y(M,g)>0$, by changing the metric g conformally, we can assume $R_g>0$(see Proposition 1.17). In particular, the conformal Laplacian

$$L_g=-\frac{4(n-1)}{n-2}\triangle_g+R_g$$

is invertible. For any $p\in M$, there exists a Green's function $0<G_p\in C^{\infty}(M\setminus\{p\})$ to L_g, i.e.,

$$L_gG_p=\delta_p \quad \text{in } M\setminus\{p\}. \tag{3.10}$$

Here δ_p denotes the Dirac delta function at p, i.e.,

$$\int_M fL_gG_p dV_g=f(p) \quad \text{for all } f\in C^{\infty}(M).$$

We accept the following theorem frequently.

Theorem 3.13. *Suppose that $3 \le n \le 5$ or M is locally conformally flat, in conformal normal coordinates at p. We have the following asymptotic expansion:*

$$G_p = r^{2-n} + A + \alpha(x), \tag{3.11}$$

where $r = \operatorname{dist}(p, x)$, where A is a real number and $\alpha = O(r)$ and $\alpha \in C^{2,\beta}$ for some $\beta > 0$.

For the proof, see Lee and Parker and Scheon-Yau.

Theorem 3.14. *Let (M, g) be a Riemannian manifold. Then for fixed $p \in M$, $\left(M \setminus \{p\}, \hat{g} = G_p^{\frac{4}{n-2}} g\right)$ is asympotically flat with $R_{\hat{g}} \equiv 0$. Also, $m_{\mathrm{ADM}}(\hat{g}) = c_n A$, where A is the constant in* (3.11).

Proof. By Theorem 3.13, in a normal coordinate, we have

$$G_p = r^{2-n} + A + \alpha(x),$$

where r is the distance from p to x, where A is a real number and $\alpha = O(r)$ and $\alpha \in C^{2,\beta}$ for some $\beta > 0$. Note that $L_g G_p = 0$ in $M \setminus \{p\}$. So $R_{\hat{g}} \equiv 0$ from the Yamabe equation. Let U be an open neighborhood at p, in which the conformal normal coordinates is defined. Without loss of generality, we assume that $U = \{(x_1, \ldots, x_n) : |x - p| = r < \rho\}$. Set $K = (M \setminus \{p\}) \setminus U$. Then K is compact in $M \setminus \{p\}$. Note that $(M \setminus \{p\}) \setminus K = U \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^n \setminus B_{1/\rho}(0)$ through the map

$$\begin{aligned} U \setminus \{p\} &\to \mathbb{R}^n \setminus B_\rho(0) \\ (x_1, \ldots, x_n) &\mapsto \left(\frac{x_1 - p_1}{r^2}, \ldots, \frac{x_n - p_n}{r^2}\right) \end{aligned}$$

where r denotes the distance between x and p.

In U, from $g_{ij}(x) = \delta_{ij} + f_{ij}$ and Taylor's expansion, we have

$$\begin{aligned} \hat{g}_{ij}(x) &= G_p(x)^{\frac{4}{n-2}} g_{ij}(x) \\ &= \left(r^{2-n} + A + \alpha(x)\right)^{\frac{4}{n-2}} (\delta_{ij} + f_{ij}) \\ &= r^{-4}\left(1 + Ar^{n-2} + \alpha(x) r^{n-2}\right)^{\frac{4}{n-2}} (\delta_{ij} + f_{ij}) \\ &= r^{-4}\left(1 + \frac{4}{n-2} A + r^{n-2}\right) \delta_{ij} + \beta_{ij}, \end{aligned}$$

where $\beta = O\left(r^{n-5}\right)$, $\partial\beta = O\left(r^{n-6}\right)$, and $\partial^2\beta = O\left(r^{n-7}\right)$.

Since $y_i = \frac{x_i - p_i}{r}$,

$$\begin{aligned} \hat{g}_{ij}(y) &= \hat{g}\left(r^2 \frac{\partial}{\partial x_i}, r^2 \frac{\partial}{\partial x_j}\right) \\ &= r^4 \hat{g}_{ij}(x) \end{aligned}$$

and

$$|y| = \frac{1}{|x - p|} = \frac{1}{r}.$$

Therefore,

$$\hat{g}_{ij}(y) = \left(1 + \frac{4}{n-2} A |y|^{2-n}\right) \delta_{ij} + |y|^{-4} \beta_{ij}\left(\frac{y}{|y|^2}\right)$$

which verifies that $(M \setminus \{p\}, \hat{g})$ is asymptotic flat (Exercise!)

Hence, the ADM mass of $(M \setminus \{p\}, \hat{g})$ is well-defined and the straightforward calculation gives

$$m_{\mathrm{ADM}}(\hat{g}) = c_n A$$

for some dimensional constant c_n. By the positive mass theorem, $A \geq 0$ and $A \equiv 0$ if and only if $(M \setminus \{p\}, \hat{g})$ is isometric to a Euclidean space. □

We are ready to prove the rest case of the Yamabe problem. This was proved by Schoen.

Theorem 3.15 (Schoen). *Let* (M, g) *be a Riemannian manifold with* $Y(M, g) \geq 0$ *and* $3 \leq \dim M \leq 5$. *If* (M, g) *is not conformally equivalent to* $(\mathbb{S}^n, g_{\mathbb{S}^n})$*, then* $Y(M, g) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$. *Hence, the Yamabe problem is solvable in this case.*

Proof. Fix $p \in M$ and let U be the conformal neighborhood of p, wehre its conformal normal coordinates is defined(see Theorem 2.19). Recall our bubble function

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}}$$

and

$$\eta(x) = \eta(r) = \begin{cases} 1 & \text{if } r < \rho_0 \\ 0 & \text{if } r \geq 2\rho_0 \end{cases}$$

and $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq c/\rho_0$ for some $\rho_0 > 0$.

By Theorem 3.13, we have

$$G = r^{2-n} + A + \alpha(x),$$

where $\alpha \in C^{2,\beta}$, $\alpha(x) = O(r)$, where $r = |x - p|$. Define the test function

$$\varphi(x) = \begin{cases} u_\varepsilon(x) & \text{if } |x| < \rho_0, \\ \varepsilon_0 (G_p(x) - \eta(x)\alpha(x)) & \text{if } \rho_0 \leq |x| < 2\rho_0 \\ \varepsilon_0 G_p(x) & \text{if } x \in M \setminus B_{2\rho_0}(p). \end{cases},$$

Here $\varepsilon_0 > 0$ and $\varepsilon_0 \ll \rho_0$ satisfying

$$\varepsilon_0 \left(\rho_0^{2-N} + A\right) = \left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}\right)^{\frac{n-2}{2}}. \tag{3.12}$$

Note that φ is Lipschitz continuous. Indeed, when $r = 2\rho_0$,

$$\varphi(x) = \varepsilon_0 G_p(x) \quad \text{since } \eta(x) = 0 \quad \text{at } r = 2\rho_0.$$

When $r = \rho_0$, by (3.12) and the asymptotic expansion of G_p,

$$\varepsilon_0 (G_p(x) - \alpha(x)) = \varepsilon_0 \left(\rho_0^{2-n} + A\right) = \left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}\right)^{\frac{n-2}{2}}.$$

We claim that this φ is the desired one. Note that

$$g_{ij} = \delta_{ij} + O\left(r^2\right), \quad \det g = 1 + O\left(r^N\right) \quad \text{and} \quad R_g = O\left(r^2\right).$$

We estimate

$$\int_M \left(\frac{4(n-1)}{n-2} |\nabla_g \varphi|^2 + R_g \varphi^2\right) dV_g$$

$$
\begin{aligned}
&= \int_{B_{\rho_0}} \left(\frac{4(n-1)}{n-2} |\nabla_g \varphi|^2 + R_g \varphi^2 \right) dV_g + \int_{M \setminus B_{\rho_0}} \left(\frac{4(n-1)}{n-2} |\nabla_g \varphi|^2 + R_g \varphi^2 \right) dV_g \\
&= I + II.
\end{aligned}
$$

If $\rho_0 \leq r \leq 2\rho_0$, we have

$$
\begin{aligned}
|\nabla_g \varphi|^2 &= \varepsilon_0^2 |\nabla_g (G_p - \eta\alpha)|^2 \\
&= \varepsilon_0^2 \left(|\nabla_g G_p|^2 + |\nabla_g (\eta\alpha)|^2 - 2 \langle \nabla_g G_p, \nabla_g (\eta\alpha) \rangle \right).
\end{aligned}
$$

So

$$
\begin{aligned}
& II \\
&= \int_{M \setminus B_{\rho_0}} \left(\frac{4(n-1)}{n-2} |\nabla_g \varphi|^2 + R_g \varphi^2 \right) dV_g \\
&= \int_{M \setminus B_{\rho_0}} \varepsilon^2 \left(\frac{4(n-1)}{n-2} |\nabla_g G_p|^2 + R_g G_p^2 \right) dV_g \\
&\quad + \varepsilon_0^2 \int_{B_{2\rho_0} \setminus B_{\rho_0}} \frac{4(n-1)}{n-2} \left(|\nabla_g (\eta\alpha)|^2 - 2 \langle \nabla_g G_p, \nabla_g (\eta\alpha) \rangle + R_g \left(\eta^2 \alpha^2 - 2\eta\alpha G_p \right) \right) dV_g \\
&= III + IV.
\end{aligned}
$$

By using $|\nabla_g (\eta\alpha)| \leq c$ and $|\nabla_g G| \leq c r^{1-n}$, we have

$$
|IV| \leq c \rho_0 \varepsilon_0^2.
$$

To estimate III, we use integration by parts to get

$$
\begin{aligned}
III &= \varepsilon_0^2 \left(\int_{M \setminus B_\rho} G_p L_g G_p dV_g + \frac{4(n-1)}{n-2} \int_{\partial B_{\rho_0}} G_p \frac{\partial G_p}{\partial \nu_g} dS_g \right) \\
&= -\varepsilon_0^2 \frac{4(n-1)}{n-2} \int_{\partial B_{\rho_0}} G_p \frac{\partial G_p}{\partial \nu_g} dS_g,
\end{aligned}
$$

where ν_g denotes the outward unit normal vector. So

$$
II \lesssim \frac{4(n-1)}{n-2} \varepsilon_0^2 \int_{\partial B_{\rho_0}} G_p \frac{\partial G_p}{\partial \nu_g} dS_g + \rho_0 \varepsilon_0^2. \tag{3.13}
$$

For the first part, as we done before using the conformal normal coordinate, we have

$$
I \lesssim \int_{B_{\rho_0}} \frac{4(n-1)}{n-2} |\nabla_g u_\varepsilon|^2 dx + c \rho_0^{6-n} \varepsilon_0^2.
$$

By taking ntegrating by parts, we have

$$
\begin{aligned}
&= \int_{\partial B_{\rho_0}} \frac{4(n-1)}{n-2} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_g} dS_g \\
&\quad - \int_{B_{\rho_0}} \frac{4(n-1)}{n-2} u_\varepsilon \triangle u_\varepsilon + c \rho_0^{6-n} \varepsilon_0^2.
\end{aligned} \tag{3.14}
$$

Recall that

$$
-\frac{4(n-1)}{n-2} \triangle u_\varepsilon = \Lambda u_\varepsilon^{\frac{n+2}{n-2}}.
$$

Combining (3.13) and (3.14), we have

$$\begin{aligned} E(\varphi) &= \frac{\int_M \left(\frac{4(n-1)}{n-2}\left|\nabla_g \varphi\right|^2 + R_g \varphi^2\right) dV_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}}} \\ &\leq \Lambda \left\|\varphi\right\|^2_{\frac{2n}{n-2}} + c\rho_0 \varepsilon_0^2 \\ &\quad + \frac{4(n-1)}{n-2}\int_{\partial B_{\rho_0}} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G_p \frac{\partial G_p}{\partial \nu_g}\right) dS_g. \end{aligned}$$

Note that

$$\frac{\partial u_\varepsilon}{\partial r} = -(n-2)\left(\frac{\varepsilon}{\varepsilon^2 + r^2}\right)^{\frac{n-2}{2}} \left(\frac{r}{\varepsilon^2 + r^2}\right)$$

and so by (3.12),

$$\begin{aligned} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} &= -(n-2)\left(\frac{\varepsilon}{\varepsilon_0^2 + \rho_0^2}\right)^{n-2}\left(\frac{\rho_0}{\varepsilon^2 + \rho_0^2}\right) \\ &= -(n-2)\,\varepsilon_0^2 \left(\rho_0^{2-n} + A\right)^2 \left(\frac{\rho_0}{\varepsilon^2 + \rho_0^2}\right) \\ &= -(n-2)\,\varepsilon_0^2 \left(\rho_0^{2-n} + A\right)^2 \frac{1}{\rho_0} \\ &= -(n-2)\,\varepsilon_0^2 \left(\rho_0^{3-n} + 2A\rho_0^{1-n} + O\left(\rho_0^{-1}\right)\right). \end{aligned} \tag{3.15}$$

Similarly, on ∂B_{ρ_0},

$$\begin{aligned} \varepsilon_0^2 G_p \frac{\partial G_p}{\partial \nu_g} &= \varepsilon_0^2 \left(\rho_0^{2-n} + A + O(\rho_0)\right)\left(-\frac{n-2}{\rho_0^{n-1}} + O(1)\right) \\ &= -(n-2)\,\varepsilon_0^2 \left(\rho_0^{3-n} + A\rho_0^{1-n} + O\left(\rho_0^{2-n}\right)\right). \end{aligned} \tag{3.16}$$

So by (3.15) and (3.16),

$$\begin{aligned} & u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G_p \frac{\partial G_p}{\partial \nu_g} \\ &= -(n-2)\,\varepsilon_0^2 \left(A\rho_0^{1-n} + O\left(\rho_0^{2-n}\right)\right) \end{aligned}$$

since $n \geq 3$. Thus,

$$\begin{aligned} & \int_{\partial B_{\rho_0}} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G_p \frac{\partial G_p}{\partial \nu_g}\right) dS_g \\ &= -(n-2)\,\varepsilon_0^2 \left(A\rho_0^{1-n} + O\left(\rho_0^{2-n}\right)\right)\omega_{n-1}\rho_0^{n-1} \\ &= -(n-2)\,\varepsilon_0^2 \omega_{n-1}\left(A + C\rho_0\varepsilon_0^2\right). \end{aligned}$$

Thus,

$$E(\varphi) \leq \Lambda \left\|\varphi\right\|^2_{\frac{2n}{n-2}} + C\rho_0\varepsilon_0^2 - (n-2)\,\omega_{n-1}\varepsilon_0^2 A + c\rho_0\varepsilon_0^2.$$

Since (M, g_0) is not conformally equivalent to $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $(M \setminus \{p\}, \hat{g})$ is not isometric to $(\mathbb{R}^n, \delta)$. Hence by the positive mass theorem, $A > 0$. Thus,

$$Y(M, g) \leq E(\varphi) < \Lambda = Y(\mathbb{S}^n, g_{\mathbb{S}^n}).$$

This completes the proof. □

Similarly, following the above proof, one can prove that if $n \geq 6$ and (M, g) is locally conformally flat, then the Yamabe problem is solvable.

4 Yamabe flow

In this chapter, we introduce the flow approach to the Yamabe problem. In the late 1980's, Richard Hamilton introduced the Yamabe flow to study the Yamabe problem.

4.1 Basic properties of the Yamabe flow

Definition 4.1. Let (M, g_0) be a compact Riemannian manifold. The *Yamabe flow* is the evolution equation of Riemannian metric $g = g(t)$ defined by

$$\begin{cases} \frac{\partial}{\partial t} g = -\left(R_g - \overline{R}_g\right) g & \text{for } t \geq 0 \\ g \mid_{t=0} = g_0, \end{cases}$$

where R_g denotes the scalar curvature of g and $\overline{R}_g$ is the average of R_g, i.e.,

$$\overline{R}_g = \frac{\int_M R_g dV_g}{\int_M dV_g}.$$

Remark. We skip writing space variable x and time variable t. The motivation of the Yamabe flow is to deform the metric $g = g(t)$ until it has a constant scalar curvature, as a result, solving the Yamabe problem.

Here we present some properties of Yamabe flow.

Proposition 4.2.

(i) *The Yamabe flow is steady if and only if the scalar curvature of g_0 is constant.*

(ii) *The Yamabe flow preserves the conformal structure, i.e., $g \in [g_0]$.*

Proof. (i) The Yamabe flow is steady, i.e., $g(t) = g_0$ for all $t \geq 0$ if and only if $\frac{\partial}{\partial t} g(t) = 0$ for all $t \geq 0$. By the definition of Yamabe flow, it is equivalent to $R_{g(t)} = \overline{R}_{g(t)}$ for all $t \geq 0$.

(ii) Suppose g is a solution of Yamabe flow for $t \in [0, T)$ for some $T > 0$. For $t \in [0, T)$, we claim that

$$g(t) = e^{-\int_0^t \left(R_{g(\tau)} - \overline{R}_{g(\tau)}\right) d\tau} g_0,$$

which is a well-defined Riemannian metric.

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -\left(R_{g(t)} - \overline{R}_{g(t)}\right) e^{-\int_0^t \left(R_{g(\tau)} - \overline{R}_{g(\tau)}\right) d\tau} g_0 \\ &= -\left(R_{g(t)} - \overline{R}_{g(t)}\right) g(t). \end{aligned}$$

□

By Proposition 4.2 (ii), we can write

$$g = u^{\frac{4}{n-2}} g_0 \tag{4.1}$$

for some positive function $u = u(t) \in C^\infty(M)$. Substituting this into the Yamabe flow, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\left(u^{\frac{4}{n-2}} g_0\right) = -\left(R_g - \overline{R}_g\right) u^{\frac{4}{n-2}} g_0 \\ &u^{\frac{4}{n-2}} g_0 \mid_{t=0} = g_0 \end{aligned}$$

if and only if

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{n-2}{4}\left(R_g - \overline{R}_g\right) u & \text{for } t \geq 0. \\ u \mid_{t=0} = 1 \end{cases} \tag{4.2}$$

This is the evolution equation for the conformal factor $u = u(t)$. Note that this is a scalar equation. Note that the Ricci flow is a system of PDE. So the solving Yamabe flow is easier than Ricci flow.

By (4.1), we have the Yamabe equation

$$-\frac{4(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u = R_g u^{\frac{n+2}{n-2}}$$

and so

$$u^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u\right) = R_g. \tag{4.3}$$

Substituting this into the first equation in (4.2), we get

$$\frac{\partial u}{\partial t} = -\frac{n-2}{4}\left[u^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u\right) - \overline{R}_g\right]u,$$

i.e.,

$$\frac{\partial}{\partial t}\left(u(t)^{\frac{n+2}{n-2}}\right) = \frac{n+2}{4}\left(\frac{4(n-1)}{n-2}\triangle_{g_0}u(t) - R_{g_0}u(t) + \overline{R}_{g(t)}u(t)^{\frac{n+2}{n-2}}\right),$$

which is a parabolic equation of u. Hence, it has a short-time existence and uniqueness. This is an basic and important proposition because it tells that we can start the flow.

Proposition 4.3. *Given any Riemannian metric g_0, there exists $T > 0$ such that there exists a unique solution $g = g(t)$ of Yamabe flow for $t \in [0, T)$ with the initial condition $g|_{t=0} = g_0$.*

Another property is the following:

Proposition 4.4. *Along the Yamabe flow, the volume of M is preserved, i.e.,*

$$\int_M dV_g = \int_M dV_{g_0} \quad \textit{for all } t \geq 0.$$

The proof is left as an exercise to the reader

Recall the previous approach of Yamabe problem. We want to find a minimizer of the Yamabe functional, which is a solution of Yamabe problem. The following proprosition tells that Yamabe flow decreases the Yamabe energy.

Proposition 4.5. *Along the Yamabe flow, we have*

$$\frac{d}{dt}E_{g_0}(u) \leq 0.$$

Proof. Recall the definition of the Yamabe energy.

$$E_{g_0}(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla_{g_0}u|^2 + R_{g_0}u^2\right)dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}}dV_{g_0}\right)^{\frac{n-2}{n}}}.$$

By Proposition 4.4, it suffices to differentiate the numerator of the Yamabe energy. Then using integration by part and (4.2), we have

$$\begin{aligned}&\frac{d}{dt}\int_M \left(\frac{4(n-1)}{n-2}|\nabla_{g_0}u|^2 + R_{g_0}u^2\right)dV_{g_0}\\ &= 2\int_M \left(\frac{4(n-1)}{n-2}\left\langle \nabla_{g_0}u, \nabla_{g_0}\frac{\partial u}{\partial t}\right\rangle + R_{g_0}u\frac{\partial u}{\partial t}\right)dV_{g_0}\end{aligned}$$

$$
\begin{aligned}
&= 2\int_M \left(-\frac{4(n-1)}{n-2}\triangle_{g_0}u + R_{g_0}u\right)\frac{\partial u}{\partial t}dV_{g_0}\\
&= 2\int_M R_g u^{\frac{n+2}{n-2}}\left[-\frac{n-2}{4}\left(R_g-\overline{R}_g\right)u\right]dV_{g_0}\\
&= -\frac{n-2}{2}\int_M R_g\left(R_g-\overline{R}_g\right)dV_g.
\end{aligned}
$$

Add $\frac{n-2}{2}\overline{R}_g\left(\int_M R_g-\overline{R}_g\right)dV_g$ in the above, although it is zero by the definition of $\overline{R}_g$. Then it is equal to

$$
= -\frac{n-2}{2}\int_M \left(R_g-\overline{R}_g\right)^2 dV_g \leq 0.
$$

So

$$
\frac{d}{dt}E_{g_0}(u) = \frac{-\frac{n-2}{2}\int_M\left(R_g-\overline{R}_g\right)^2dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}} \leq 0. \tag{4.4}
$$

This completes the proof of Proposition 4.5. □

Suppose that the solution g of Yamabe flow exists on $[0,\infty)\times M$. Then for any $T>0$, we can integrate (4.4) from 0 to T to get

$$
E_{g_0}(u(T)) - E_{g_0}(u(0)) = \frac{-\frac{n-2}{2}\int_0^T\int_M\left(R_g-\overline{R}_g\right)^2dV_gdt}{\left(\int_M dV_{g_0}\right)^{\frac{n-2}{n}}}.
$$

This implies that

$$
\begin{aligned}
\frac{n-2}{2}\int_0^T\int_M\left(R_g-\overline{R}_g\right)dV_gdt &= \left(E_{g_0}(1)-E_{g_0}(u(T))\right)\left(\int_M dV_{g_0}\right)^{\frac{n-2}{n}}\\
&\leq \left(E_{g_0}(1)-Y(M,g_0)\right)\left(\int_M dV_{g_0}\right)^{\frac{n-2}{n}}\\
&\leq C.
\end{aligned}
$$

Hence the upper bound does not depend on T. Letting $T\to\infty$, we have

$$
\frac{n-2}{2}\int_0^\infty\int_M\left(R_g-\overline{R}_g\right)^2dV_gdt \leq C
$$

for some uniform constant C depending only on g_0. So

$$
\liminf_{t\to\infty}\int_M\left(R_g-\overline{R}_g\right)^2dV_g = 0.
$$

Proposition 4.6. *If* $g = u^{\frac{4}{n-2}}g_0$*, then*

(i) $\langle\nabla_g f_1,\nabla_g f_2\rangle_g = u^{-\frac{4}{n-2}}\langle\nabla_{g_0}f_1,\nabla_{g_0}f_2\rangle_{g_0}$.

(ii) $\triangle_g f = u^{-\frac{n+2}{n-2}}\left(n\triangle_{g_0}f + 2\langle\nabla_{g_0}f,\nabla_{g_0}u\rangle_{g_0}\right)$.

Proof. (i) In a local coordinate, we have

$$
\nabla_g f = g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}
$$

$$= u^{-\frac{4}{n-2}} g_0^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$$
$$= u^{-\frac{4}{n-2}} \nabla_{g_0} f.$$

So

$$\langle \nabla_g f_1, \nabla_g f_2 \rangle_g = \left\langle u^{-\frac{4}{n-2}} \nabla_{g_0} f_1, u^{-\frac{4}{n-2}} \nabla_{g_0} f_2 \right\rangle$$
$$= u^{-\frac{4}{n-2}} \langle \nabla_{g_0} f_1, \nabla g_0 f_2 \rangle_{g_0}$$

since $\langle \cdot, \cdot \rangle_g = u^{\frac{4}{n-2}} \langle \cdot, \cdot \rangle_{g_0}$.

(ii) In a local coordinate, recall that

$$\triangle_g f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x_j} \right).$$

Since $g = u^{-\frac{4}{n-2}} g_0$, $g^{ij} = u^{-\frac{4}{n-2}} (g_0)^{ij}$. Also, we have $\det g = u^{\frac{4n}{n-2}} \det g_0$. So

$$\triangle_g f = \frac{1}{\sqrt{\det (g_0)} u^{\frac{2n}{n-2}}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g_0} u^{\frac{2n}{n-2}} u^{-\frac{4}{n-2}} (g_0)^{ij} \frac{\partial f}{\partial x_j} \right)$$
$$= \frac{1}{\sqrt{\det (g_0)} u^{\frac{2n}{n-2}}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g_0} u^2 (g_0)^{ij} \frac{\partial f}{\partial x_j} \right)$$
$$= \frac{u^2}{u^{\frac{2n}{n-2}}} \triangle_{g_0} f + \frac{2u}{u^{\frac{2n}{n-2}}} (g_0)^{ij} \frac{\partial u}{\partial x_i} \frac{\partial f}{\partial x_j}$$
$$= u^{-\frac{n+2}{n-2}} \left(n \triangle_{g_0} f + 2 \langle \nabla_{g_0} f, \nabla_{g_0} u \rangle_{g_0} \right).$$

This completes the proof. □

Proposition 4.7. *Along the Yamabe flow, the scalar curvature satisfies the evolution equation:*

$$\frac{\partial}{\partial t} R_g = (n-1) \triangle_g R_g + R_g \left(R_g - \overline{R}_g \right).$$

Proof. Differentiate (4.3) with respect to t. Then

$$\frac{\partial}{\partial t} R_g = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} \frac{\partial u}{\partial t} + R_{g_0} \frac{\partial u}{\partial t} \right)$$
$$- \frac{n+2}{n-2} u^{-\frac{n+2}{n-2}-1} \frac{\partial u}{\partial t} \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u \right)$$
$$= u^{-\frac{n+2}{n-2}} \left[-\frac{4(n-1)}{n-2} \triangle_{g_0} \left(-\frac{n-2}{4} \left(R_g - \overline{R}_g \right) u \right) + R_{g_0} \left(-\frac{n-2}{4} \left(R_g - \overline{R}_g \right) u \right) \right]$$
$$- \frac{n+2}{n-2} u^{-\frac{n+2}{n-2}-1} \left(-\frac{n-2}{4} \left(R_g - \overline{R}_g \right) u \right) \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u \right)$$
$$= -\frac{n-2}{4} u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} \left(-\frac{n-2}{4} \left(R_g - \overline{R}_g \right) u \right) + R_{g_0} \left(-\frac{n-2}{4} \left(R_g - \overline{R}_g \right) u \right) \right)$$
$$+ \frac{n+2}{4} u^{-\frac{n+2}{n-2}} \left(R_g - \overline{R}_g \right) \left(-\frac{4(n-1)}{n-2} \triangle_{g_0} u + R_{g_0} u \right)$$
$$= -\frac{n-2}{4} u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \left(\left(R_g - \overline{R}_g \right) \triangle_{g_0} u + u \triangle_{g_0} R_g \right. \right.$$

$$
\begin{aligned}
&\quad +2\left\langle \nabla_{g_0} R_g, \nabla_{g_0} u\right\rangle_{g_0}\Big) + R_{g_0}\left(R_g - \overline{R}_g\right) u\Big) \\
&\quad +\frac{n+2}{4} u^{-\frac{n+2}{n-2}}\left(R_g - \overline{R}_g\right)\left(-\frac{4(n-1)}{n-2}\triangle_{g_0} u + R_{g_0} u\right) \\
&= \left(\frac{n+2}{4} - \frac{n-2}{4}\right)\left(R_g - \overline{R}_g\right) u^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\triangle_{g_0} u + R_{g_0} u\right) \\
&\quad + (n-1) u^{-\frac{n+2}{n-2}}\left(u\triangle_{g_0} R_g + 2\left\langle \nabla_{g_0} R_g, \nabla_{g_0} u\right\rangle_{g_0}\right) \\
&= \left(R_g - \overline{R}_g\right) R_g + (n-1) u^{-\frac{n+2}{n-2}}\left(u\triangle_{g_0} R_g + 2\left\langle \nabla_{g_0} R_g, \nabla_{g_0} u\right\rangle_{g_0}\right).
\end{aligned}
$$

By Proposition 4.6, we are done. □

Remark. We can also compute the evolution equation of the Ricci curvature, etc,. since we know $g = u^{\frac{4}{n-2}} g_0$ and we know how Ric_g and Ric_{g_0} are related. Now differentiate Ric_g with respect to t and use the evolution equation for $\frac{\partial u}{\partial t}$.

Recall, we have proved that

$$
\frac{d}{dt} E(u(t)) = -\frac{n-2}{2} \frac{\int_M \left(R_g - \overline{R}_g\right)^2 dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}},
$$

where $g = g(t) = u(t)^{\frac{4}{n-2}} g_0$ is the solution of Yamabe flow. We can also use Proposition 4.7 to prove this.

Another proof of Proposition 4.5. Recall,

$$
E(u(t)) = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}},
$$

where $g(t) = g(t) = u(t)^{\frac{4}{n-2}} g_0$. By Proposition 4.7, we have

$$
\begin{aligned}
\frac{d}{dt}\int_M R_g dV_g &= \int_M \frac{\partial R_g}{\partial t} dV_g + \int_M R_g \frac{\partial}{\partial t} dV_g \qquad (4.5) \\
&= \int_M \left[(n-1)\triangle_g R_g + R_g\left(R_g - \overline{R}_g\right)\right] dV_g \\
&\quad + \int_M R_g\left(-\frac{n}{2}\left(R_g - \overline{R}_g\right)\right) dV_g
\end{aligned}
$$

since

$$
\frac{\partial}{\partial t} dV_g = -\frac{n}{2}\left(R_g - \overline{R}_g\right) dV_g.
$$

By Stokes' theorem, we get

$$
\begin{aligned}
&= \left(1 - \frac{n}{2}\right)\int_M R_g\left(R_g - \overline{R}_g\right) dV_g \\
&= -\frac{n-2}{2}\int_M R_g\left(R_g - \overline{R}_g\right) dV_g + \frac{n-2}{2}\overline{R}_g \int_M \left(R_g - \overline{R}_g\right) dV_g \\
&= -\frac{n-2}{2}\int_M \left(R_g - \overline{R}_g\right)^2 dV_g \le 0.
\end{aligned}
$$

This completes the proof. □

Corollary 4.8. *$t \mapsto \overline{R}_{g(t)}$ is a nonincreasing function.*

Proof. From (4.5) and Proposition 4.4, we have

$$\frac{d}{dt}\overline{R}_g = \frac{d}{dt}\left(\frac{\int_M R_g dV_g}{\int_M dV_g}\right) = \frac{-\frac{n-2}{2}\int_M \left(R_g - \overline{R}_g\right)^2 dV_g}{\int_M dV_g} \leq 0. \qquad \square$$

Note that the function is bounded below. To see this, it follows from the definition of Yamabe energy

$$\begin{aligned}\overline{R}_g &= \frac{\int_M R_{g(t)} dV_{g(t)}}{\int_M dV_{g(t)}} = \frac{\int_M R_{g(t)} dV_{g(t)}}{\left(\int_M dV_{g(t)}\right)^{\frac{n-2}{n}} \left(\int_M dV_{g(t)}\right)^{\frac{2}{n}}} \\ &= \frac{E\left(u\left(t\right)\right)}{\left(\int_M dV_{g_0}\right)^{\frac{2}{n}}} \geq \frac{Y\left(M, g_0\right)}{\left(\int_M dV_{g_0}\right)^{\frac{2}{n}}}.\end{aligned}$$

By Corolloary 4.8, $\lim_{t\to\infty} \overline{R}_{g(t)}$ exists. Recall that we have proved

$$\liminf_{t\to\infty} \int_M \left(R_{g(t)} - \overline{R_{g(t)}}\right)^2 dV_{g(t)} = 0.$$

If $g\left(t\right) \to g_\infty$ as $t \to \infty$, then by Fatou's lemma, we have

$$R_{g_\infty} = \overline{R}_{g_\infty} = \lim_{t\to\infty} \overline{R_{g(t)}},$$

which is constant. Here we again note that this method does not give an information on minimizer. We don't know whether it is a minimizer of the Yamabe energy.

4.2 The Yamabe flow when $Y(M, g_0) < 0$

We focus on $Y\left(M, g_0\right) < 0$. We are going to prove the long-time existence and the convergence of the Yamabe flow. One might expect this result since we can prove the original Yamabe problem easily in this case. By changing the metric conformally, we can assume that $R_{g_0} < 0$.

Proposition 4.9. *If $R_{g_0} < 0$, then $R_g \leq \max R_{g_0}$ along the Yamabe flow.*

Proof. We try to use the maximum principle. For $\varepsilon > 0$, we consider

$$F_\varepsilon\left(x, t\right) = R_{g(t)} - \varepsilon,$$

which is a smooth function on $M \times [0, T)$. Choose ε to be small enough such that

$$\max R_{g_0} + \varepsilon < 0. \tag{4.6}$$

We claim that

$$F_\varepsilon < \max R_{g_0}.$$

Suppose not. Then there exists $(x_0, t_0) \in M \times [0, T)$ such that

$$F_\varepsilon\left(x_0, t_0\right) \geq \max R_{g_0}. \tag{4.7}$$

Then $t_0 > 0$ since if $t_0 = 0$, then

$$F\left(x_0, 0\right) = R_{g_0}\left(x_0\right) - \varepsilon \leq \max R_{g_0} - \varepsilon < \max R_{g_0},$$

which leads a contradiction.

Assume t_0 to be the smallest to which satisfies (4.7), i.e., for any $t < t_0$,

$$F_\varepsilon(x, t) < \max R_{g_0}. \tag{4.8}$$

By continuity, letting $t \to t_0-$, we have

$$F_\varepsilon(x, t_0) \leq \max R_{g_0} \quad \text{for all } x \in M. \tag{4.9}$$

Combining (4.7) and (4.9), we see that $F_\varepsilon(x_0, t_0) = \max R_{g_0}$. By definition of F_ε, we have

$$R_{g_0(t_0)}(x_0) = \max R_{g_0} + \varepsilon. \tag{4.10}$$

By the same reason, we also have

$$F_\varepsilon(x_0, t_0) = \max_{x \in M} F_\varepsilon(x, t_0). \tag{4.11}$$

At (x_0, t_0), we have

$$0 \leq \frac{\partial}{\partial t} F_\varepsilon(x_0, t_0)$$

by (4.8) and (4.9). By the definition of F_ε, (4.10), and (4.11), we have

$$\begin{aligned} \frac{\partial}{\partial t} F_\varepsilon(x_0, t_0) &= \frac{\partial}{\partial t} R_{g(t_0)} = (n-1) \triangle_{g(t_0)} R_{g(t_0)} + R_{g(t_0)} \left(R_{g(t_0)} - \overline{R}_{g(t_0)}\right) \\ &\leq 0 + \left(\max_M R_{g_0} + \varepsilon\right)\left(\max_M R_{g_0} + \varepsilon - \overline{R}_{g(t_0)}\right) \\ &< 0 \end{aligned}$$

since (4.6) and

$$\max_M R_{g_0} + \varepsilon - \overline{R}_{g(t_0)} \geq \max_M R_{g_0} + \varepsilon - \overline{R}_{g_0} > 0$$

holds. Here we used the nondecreasing property of $\overline{R}_g$, which leads a contradiction. This proves the claim. Letting $\varepsilon \to 0$, we get $R_g \leq \max R_{g_0}$.This completes the proof. □

Similarly, we have

Proposition 4.10. *If $R_{g_0} < 0$, then*

$$\min_M R_{g_0} \leq R_g$$

along the Yamabe flow.

Proof. For $\varepsilon > 0$, define

$$F_\varepsilon(x, t) = R_{g(t)}(x) + \varepsilon(t+1),$$

which is smooth on $M \times [0, T)$. Then we claim that

$$F_\varepsilon > \min_M R_{g_0}.$$

Suppose not. Then there exists (x_0, t_0) such that

$$F_\varepsilon(x_0, t_0) \leq \min_M R_{g_0}.$$

Then following the argument in the proof of Proposition 4.9, we can show that $t_0 > 0$. Then choose the smallest t_0 so that

$$F_\varepsilon(x_0, t_0) = \min_M R_{g_0}$$

which implies that

$$R_{g(t_0)}(x_0) = \min_M R_{g_0} - \varepsilon.$$

Then at (x_0, t_0), we have

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial t} F_\varepsilon (x_0, t_0) = \varepsilon + \frac{\partial}{\partial t} R_{g(t_0)} \\ &= \varepsilon + (n-1) \triangle_{g(t_0)} R_{g(t_0)} + R_{g(t_0)} \left(R_{g(t_0)} - \overline{R}_{g(t_0)} \right). \end{aligned}$$

Since R_g attains minimum at (x_0, t_0), we have $\triangle_{g(t_0)} R_{g(t_0)}(x_0) \geq 0$. From Proposition 4.9, we have $R_g \leq \max R_{g_0} < 0$, and so $R_g(t_0) < 0$ and hence

$$R_{g(t_0)}(R_{g(t_0)} - \overline{R}_{g(t_0)}) \geq 0.$$

This leads a contradiction. □

Combining these two propositions, we have

$$\min_M R_{g_0} \leq R_g \leq \max_M R_{g_0} \tag{4.12}$$

along the Yamabe flow.

Proposition 4.11. *If $Y(M, g_0) < 0$, then the Yamabe flow exists for all $t \geq 0$.*

Proof. We know that the Yamabe flow is equivalent to solving the evolution equation of the conformal factor:

$$\frac{\partial}{\partial t} u(t) = -\frac{n-2}{4} \left(R_{g(t)} - \overline{R}_{g(t)} \right) u(t).$$

By (4.12),

$$\begin{aligned} \frac{\partial}{\partial t} u(t) &= -\frac{n-2}{4} \left(R_{g(t)} - \overline{R}_{g(t)} \right) u(t) \\ &\leq \frac{n-2}{4} \left(\max_M R_0 - \min_M R_{g_0} \right) u(t). \end{aligned}$$

So

$$u(t) \leq \exp\left(\frac{n-2}{4} \left(\max_M R_{g_0} - \min_M R_{g_0} \right) \right) T$$

for all $t \in [0, T]$ and for any $T > 0$. This implies that $u(t) \in C^0(M \times [0, T])$. Since $u(t)$ is a solution of the parabolic equation, $u(t) \in C^\infty(M \times [0, T])$. □

Proposition 4.12. *We have*

$$\sup_M u(t)^{\frac{n+2}{n-2}} \leq \max \left\{ \sup_M u(0)^{\frac{n+2}{n-2}}, \left(\frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right)^{\frac{n+2}{4}} \right\} =: c_0.$$

Proof. For $\varepsilon > 0$, define

$$F_\varepsilon(x, t) = u(x, t)^{\frac{n+2}{n-2}} - \varepsilon(1 + t).$$

We claim that $F_\varepsilon(x, t) < c_0$. Suppose not. Then there exists $(x_0, t_0) \in M \times [0, \infty)$ such that

$$F_\varepsilon(x_0, t_0) \geq c_0. \tag{4.13}$$

Note that $t_0 > 0$ since

$$\begin{aligned} F_\varepsilon(x,0) &= u(0)^{\frac{n+2}{n-2}} - \varepsilon \\ &\le \sup_M u(0)^{\frac{n+2}{n-2}} \le c_0. \end{aligned}$$

Choose the smallest t_0 such that (4.13) is satisfied, i.e.,

$$F_\varepsilon(x,t) < c_0 \quad \text{for all } t < t_0. \tag{4.14}$$

So combining (4.13) and (4.14), we have

$$F(x_0,t_0) = \max_M F(x,t_0). \tag{4.15}$$

By (4.13) and the definition of F_ε, we have

$$u(x_0,t_0)^{\frac{n+2}{n-2}} \ge \varepsilon + \left(\frac{\min R_{g_0}}{\max R_{g_0}}\right)^{\frac{n+2}{4}}. \tag{4.16}$$

At (x_0,t_0), by (4.13) and (4.14), we have

$$0 \le \frac{\partial}{\partial t} F_\varepsilon(x_0,t_0) = \frac{\partial}{\partial t} u(t)^{\frac{n+2}{n-2}} - \varepsilon.$$

Using the evolution equation of the conformal factor and the Yamabe equation, we have

$$\begin{aligned} &= -\varepsilon + \frac{n+2}{n-2} u(x_0,t_0)^{\frac{n+2}{n-2}-1} \frac{\partial u}{\partial t}(x_0,t_0) \\ &= -\varepsilon + \frac{n+2}{4} u(x_0,t_0)^{\frac{n+2}{n-2}} \left(\overline{R}_{g(t_0)} - R_{g(t_0)}\right) \\ &= -\varepsilon + \frac{n+2}{4} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u - R_{g_0} u + \overline{R}_{g(t)} u^{\frac{n+2}{n-2}}\right) \\ &\le -\varepsilon + \frac{n+2}{4} \left(\max R_{g_0} u^{\frac{n+2}{n-2}} - \max R_{g_0}\right) < 0. \end{aligned}$$

So we are done. □

Similarly, we can prove the following proposition.

Proposition 4.13. *We have*

$$\inf_M u(t)^{\frac{n+2}{n-2}} \ge c_1$$

for some constant c_1.

Based on these propositions, we have $c_1 \le u(t) \le c_0$ for all $t \in [0,\infty)$ and also $u(t) \in C^\infty([0,\infty)] \times M)$.

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