Bilinear estimates in BMO and the Navier-Stokes equations

– This presentation is based on H. Kozono, Y. Taniuchi (2000, Math Z.)

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Topics in Fluid mechanics I

Introduction

In this presentation, we consider the following problem (NS):

$$\begin{cases} \partial_t u - \triangle u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{NS}$$

where $u(x,t) = \left(u^1(x,t), \ldots, u^n(x,t)\right)$ and $p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, respectively, while u_0 is the given initial velocity vector.

We consider two type of solutions of (NS)

Definition (Leray-Hopf Weak solution)

Let $u_0 \in L^2_\sigma$. A function $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$ is said to be a *Leray-Hopf weak solution* of (NS) on $(0,T)$ if

1.
$$\int_0^T [-(u(t), v_t(t)) + (\nabla u(t), \nabla v(t)) + ((u(t)\cdot\nabla)u(t), v(t))]\, dt = (u_0, v(0)) + \int_0^T (f(t), v(t))\, dt \tag{1}$$

for all $v \in C_0^\infty([0,T)\times\mathbb{R}^n)^n$ with $\operatorname{div} v = 0$.

2. u is weakly continuous in L^2_σ on $[0,T)$.

The existence of Leray-Hopf weak solution is well-known: for an arbitrary $u_0 \in L^2$, (NS) possess a weak solution $u(t)$ on $[0,T]$ for all $T > 0$ (Leray (1934) / Hopf for a bounded domain (1951) when $n = 2, 3$).

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   for all $v \in C_0^\infty([0,T)\times\mathbb{R}^n)^n$ with $\operatorname{div} v = 0$.

2. u is weakly continuous in L^2_σ on $[0,T)$.

The existence of Leray-Hopf weak solution is well-known: for an arbitrary $u_0 \in L^2$, (NS) possess a weak solution $u(t)$ on $[0,T]$ for all $T>0$ (Leray (1934) / Hopf for a bounded domain (1951) when $n=2,3$).

The uniqueness and regularity of weak solutions have been the most outstanding open questions in the mathematical fluid mechanics and are closely related to one of the seven Clay Millennium Problems.

Definition (J.-L. Lions (1969))

Let $u_0 \in L^2_\sigma$. A measurable function u on $\mathbb{R}^n \times (0,T)$ is called a *weak solution* of (NS) on $(0,T)$ if $u \in L^\infty\left(0,T;L^2_\sigma\right) \cap L^2\left(0,T;H^1_\sigma\right)$ and the following hold:

1. $u(t)$ is continuous on $[0,T]$ in the weak topology of L^2_σ;
2. we have

$$\int_s^t \{-(u,\partial_\tau \Phi) + (\nabla u, \nabla \Phi) + ((u\cdot\nabla)u,\Phi)\}\,d\tau = -(u(t),\Phi(t)) + (u(s),\Phi(s)) \tag{2}$$

for every $0 \le s \le t < T$ and every $\Phi \in H^1\left((s,t);H^1_\sigma \cap L^n\right)$.

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Remark

If (i) $2 \le n \le 4$ or (ii) Ω is a bounded domain or exterior domain or (iii) $\mathbb{R}^n$, then every Leray-Hopf weak solution is a Lion's weak solution and vice versa (see Masuda (1984) and Giga (1986)).

Introduction

Definition (Strong solution)

Let $u_0 \in H^s_\sigma$ for $s > \frac{n}{2} - 1$. A measurable function u on $\mathbb{R}^n \times (0, T)$ is called a *strong solution of* (NS) in the class $CL_s\,(0, T)$ if

1. $u \in C\left([0, T); H^s_\sigma\right) \cap C^1\left((0, T)\,; H^s_\sigma\right) \cap C\left((0, T)\,; H^{s+2}_\sigma\right)$;
2. u satisfies (NS) with some distribution p such that $\nabla p \in C\left((0, T)\,; H^s\right)$.

The existence of solution of (NS) in this class is well-known. See Fujita-Kato(1964), Kato(1984) and Giga(1986).

Introduction

Theorem

Assume that $u_0 \in L^2_\sigma$. Let u and v be weak solutions of (NS) *satisfying the energy inequality. Suppose in addition that $v \in L^r(0,T;L^q)$ for some q and r satisfying*

$$\frac{2}{r} + \frac{n}{q} = 1, \quad n < q \le \infty, \quad \text{and } 2 \le r < \infty. \tag{Se}$$

Then $u = v$ on $[0,T]$.

The class (Se) is important from a viewpoint of scaling invariance:

$$\|u_\lambda\|_{L^r(0,\infty;L^q)} = \|u\|_{L^r(0,\infty;L^q)}$$

holds for all $\lambda > 0$ if and only if $\frac{2}{r} + \frac{n}{q} = 1$. Here

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

for $\lambda > 0$.

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Theorem

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$$\frac{2}{r} + \frac{n}{q} = 1, \quad n < q \leq \infty, \quad \text{and } 2 \leq r < \infty. \tag{Se}$$

Then $u = v$ on $[0,T]$.

- Lions-Prodi (1959), Prodi (1959); uniqueness theorem when $n = 2$.
- Foias (1961); $\Omega = \mathbb{R}^n$ with $2/r + n/q < 1, \quad n < q$.
- Serrin (1962,1963); general domain Ω in $\mathbb{R}^n$ ($n = 2,3,4$) with $2/r + n/q \leq 1$.
- Masuda (1984); removed the restriction on dimension n.
- Escauriaza-Seregin-Šverák (2003); $q = n = 3$ and $r = \infty$.

Kozono-Taniuchi proved the following result:

Theorem (Kozono-Taniuchi (2000))

1. *Let* $u_0 \in L^2_\sigma$ *and let* u, v *be two weak solutions of* (NS) *on* $(0,T)$*. Suppose that*
$$u \in L^2(0,T;\mathrm{BMO}) \tag{3}$$
and that v *satisfies the energy inequality*
$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(\tau)\|_2^2\, d\tau \le \|u_0\|_2^2, \quad 0 < t < T. \tag{4}$$
Then we have $u = v$ *on* $[0,T]$*.*
2. *Let* $u_0 \in L^2_\sigma$ *and let* u *be a weak solution with the additional property* (13)*. Then for every* $0 < \varepsilon < T$*,* u *is actually a strong solution of* (NS) *in the class* $CL_s(\varepsilon, T)$ *for* $s > \frac{n}{2} - 1$*.*

The theorem of Kozono-Taniuchi is an extension of Serrin-Masuda's criterion since it is larger than the marginal case $L^2(0,T;L^\infty)$:

$$\frac{2}{r}+\frac{n}{q}=1, \quad n<q\le\infty, \quad \text{and } 2\le r<\infty. \tag{Se}$$

$$u\in L^2\left(0,T;\mathrm{BMO}\right) \tag{KT}$$

since $L^\infty\subset\mathrm{BMO}$. We will define the space BMO in later.

Also, Kozono-Taniuchi gives a regularity criteria in terms of vorticity and deformation tensor.

Theorem

Let $u_0 \in L^2_\sigma$. *Suppose that* u *is a weak solution of* (NS) *on* $(0, T)$. *If either*

$$\operatorname{curl} u \in L^1(0, T; \mathrm{BMO})$$

or

$$\operatorname{Def} u \in L^1(0, T; \mathrm{BMO})$$

holds, then for every $0 < \varepsilon < T$, u *is actually a strong solution of* (NS) *in the class* $CL_s(\varepsilon, T)$ *for* $s > \frac{n}{2} - 1$.

Some results

Theorem (Kozono-Taniuchi)

Let $s > \frac{n}{2} - 1$ *and let* $u_0 \in H^s_\sigma$*. Suppose that* u *is a strong solution of* (NS) *in the class* $CL_s(0, T)$*. If*

$$\int_{\varepsilon_0}^{T} \|u(t)\|_{\mathrm{BMO}}^2 \, dt < \infty \quad \text{for some } 0 < \varepsilon_0 < T,$$

then u *can be continued to the strong solution in the class* $CL_s(0, T')$ *for some* $T' > T$*.*

As a corollary, we obtain a blow-up result.

Corollary (Blow-up result)

Let u *be a strong solution of* (NS) *in the class* $CL_s(0, T)$ *for* $s > \frac{n}{2} - 1$*. Suppose that* T *is maximal, i.e.,* u *cannot be continued in the class* $CL_s(0, T')$ *for any* $T' > T$*. Then*

$$\int_{\varepsilon}^{T} \|u(t)\|_{\mathrm{BMO}}^2 \, dt = \infty \quad \text{for any } 0 < \varepsilon < T.$$

In particular, we have

$$\limsup_{t \to T-} \|u(t)\|_{\mathrm{BMO}} = \infty.$$

Some results

Theorem (Kozono-Taniuchi)

Let $s > \frac{n}{2} - 1$*. Suppose that* u *is a strong solution of* (NS) *in the class* $CL_s(0,T)$*. If either*

$$\int_{\varepsilon_0}^{T} \|\mathrm{curl}\, u(t)\|_{\mathrm{BMO}}\, dt < \infty \quad \text{or} \quad \int_{\varepsilon_0}^{T} \|\mathrm{Def}\, u(t)\|_{\mathrm{BMO}}\, dt < \infty$$

holds for some $0 < \varepsilon_0 < T$*, then* u *can be continued to the strong solution in the class* $CL_s(0,T')$ *for some* $T' > T$*.*

As a corollary, we obtain a blow-up result.

Corollary (Blow-up result)

Suppose that u *is a strong solution of* (NS) *in the class* $CL_s(0,T)$ *for* $s > n/2 - 1$*. Assume that* T *is maximal in the same sense as before. Then both*

$$\int_{\varepsilon_\varepsilon}^{T} \|\mathrm{curl}\, u(t)\|_{\mathrm{BMO}}\, dt = \infty \quad \text{and} \quad \int_{\varepsilon_\varepsilon}^{T} \|\mathrm{Def}\, u(t)\|_{\mathrm{BMO}}\, dt = \infty$$

hold for all $0 < \varepsilon < T$*.*

Remarks

- In $\mathbb{R}^3$, Beale-Kato-Majda(1984) considered the following statement: if

$$\int_0^T \|\mathrm{curl}\, u(t)\|_{L^\infty}\, dt < \infty$$

then $u(t)$ can never break down its regularity at $t = T$ for incompressible Euler equation. The same assertion holds even for (NS). This papers extend this result to the marginal space BMO.

- Beirão da Veiga proved the regularity criterion in the class $\nabla u \in L^r(0, T; L^q)$ for $2/r + n/q = 2$ with $n/2 < q < \infty$ and $1 < r < \infty$. Kozono-Taniuchi covers the borderline case $q = \infty$ and $r = 1$.

Contents

Notations

- the standard Sobolev space
$$W^{k,q}(\mathbb{R}^n), \quad H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$$
- the Bessel potential space
$$L^{\gamma,q}(\mathbb{R}^n) = \{(I-\triangle)^{\gamma/2} f : f \in L^q(\mathbb{R}^n)\}, \quad H^\gamma(\mathbb{R}^n) = L^{\gamma,2}(\mathbb{R}^n)$$
- $L^{\gamma,q}_\sigma(\mathbb{R}^n) = \{u \in L^{\gamma,q}(\mathbb{R}^n) : \operatorname{div} u = 0\}$, $H^\gamma_\sigma = L^{\gamma,2}_\sigma(\mathbb{R}^n)$.
- $C_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact supports.
- $C_{0,\sigma}^\infty(\mathbb{R}^n)$ the space of smooth vector fields with divergence-free with compact supports.
- for a vector field $u : \mathbb{R}^n \to \mathbb{R}^n$, we write $u = (u^1, \ldots, u^n)$ and
$$\operatorname{curl} u = \left(D_j u^k - D_k u^j\right)_{1 \le j,k \le n}, \quad \operatorname{Def} u = \left(D_j u^k + D_k u^j\right)_{1 \le j,k \le n}.$$
- $(\cdot,\cdot)$ duality pairing between L^r and $L^{r'}$.
- $X \hookrightarrow Y$ means X is continuously embedded in Y.

We drop $\mathbb{R}^n$ if it is ambient.

The Hardy space is a good replacement of L^1 in the theory of partial differential equation. For example, consider the case of Poission equation

$$\triangle u = f \quad \text{in } B_1, \quad u = 0 \quad \text{on } \partial B_1 \quad \text{with } f \in L^1(B_1).$$

Then in general the solution $D^2 u$ does not in L^1. For example, consider the case $\mathbb{R}^2$ and let

$$u(x) = \log\log\left(e|x|^{-1}\right)$$

Then

$$\triangle u = -\frac{1}{|x|^2 \log^2\left(e|x|^{-1}\right)}.$$

Since

$$\int_0^1 \frac{1}{r\log^2(er^{-1})}dr < \infty,$$

we see that $\triangle u \in L^1$.

Motivation of Hardy space

However, $u \notin W^{2,1}(B_1)$. Write $|x| = r$. Since

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta \frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 u}{\partial y^2},$$

we see that for sufficiently small r,

$$\left|D^2 u\right| \geq \frac{\partial^2 u}{\partial r^2} = \frac{\log\left(er^{-1}\right) - 1}{r^2 \log^2\left(er^{-1}\right)} \geq \frac{1}{2r^2 \log\left(er^{-1}\right)}.$$

Since $\int_0^\varepsilon \frac{1}{r\log\left(er^{-1}\right)} dr = \infty$ for every $\varepsilon \in (0,1]$, we have $\int_{B_1} \left|D^2 u\right| dx = \infty$.

Let $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}}$, where $c_n = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}$. Whenever f is a bounded distribution($f * \Phi \in L^\infty$ whenever $\Phi \in \mathcal{S}$), we define $u(x,t) = (f * P_t)(x)$.

Definition (Hardy space)

Let $0 < p \leq \infty$. We say that a bounded distribution f is in $\mathcal{H}^p$ if $u^* \in L^p(\mathbb{R}^n)$. Here

$$u^*(x) = \sup_{|x-y|\leq t} |u(y,t)|.$$

When $p \geq 1$, its norm is defined by

$$\|f\|_{\mathcal{H}^p} = \|u^*\|_{L^p}.$$

Remark

The following are equivalent:

1. $f \in \mathcal{H}^p$;
2. There exists a $\Phi \in \mathcal{S}$ with $\int \Phi \neq 0$ so that $M_\Phi f \in L^p(\mathbb{R}^n)$, where

$$M_\Phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|.$$

Remark

1. When $1 < p < \infty$, $\mathcal{H}^p = L^p$ and $\mathcal{H}^1 \subset L^1$ but not the converse.
2. Although L^1 has no weak compactness result, we have a weak compactness result in $\mathcal{H}^1$.

Definition

A locally integrable funciton f is in $\mathrm{BMO}(\mathbb{R}^n)$ if the inequality

$$\frac{1}{|B|}\int_B |f(x) - f_B|dx \leq A \tag{5}$$

holds for all balls B. Here $f_B = |B|^{-1}\int_B f dx$ denotes the mean value of f over the ball B.
The smallest bound A for which (5) is satisfied is then taken to be the norm of f in this space, and is denoted by $\|f\|_{\mathrm{BMO}}$.

Remark

- It is easy to see that $L^\infty \subset \mathrm{BMO}(\mathbb{R}^n)$.
- A typical example of BMO function is $\log|x|$ on $\mathbb{R}$. Note that this function is not bounded.
- The space BMO is a natural substitute of L^∞ in the theory of singular integrals.

Hardy space and BMO

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- The space BMO is a natural substitute of L^∞ in the theory of singular integrals.

It is well-known that $W^{1,n}(\mathbb{R}^n)$ is embedded in $L^q(\mathbb{R}^n)$ for any $n \leq q < \infty$. Here q cannot be ∞ due to some counterexample.

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It is well-known that $W^{1,n}(\mathbb{R}^n)$ is embedded in $L^q(\mathbb{R}^n)$ for any $n \leq q < \infty$. Here q cannot be ∞ due to some counterexample.

Proposition

$W^{1,n}(\mathbb{R}^n)$ is embedded in $\mathrm{BMO}(\mathbb{R}^n)$. In general, if $1 < p < \infty$ and $\gamma > 0$ with $\gamma p = n$, then $L^{\gamma,p}(\mathbb{R}^n)$ is embedded in $\mathrm{BMO}(\mathbb{R}^n)$.

$W^{1,n}(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n)$ can be proved by using the Poincaré inequality. The general case requires hard computation.

Proposition

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Definition (Strong solution)

Let $u_0 \in H^s_\sigma$ for $s > n/2 - 1$. A measurable function u on $\mathbb{R}^n \times (0,T)$ is called a *strong solution of* (NS) in the class $CL_s\,(0,T)$ if

1. $u \in C\left([0,T); H^s_\sigma\right) \cap C^1\left((0,T)\,; H^s_\sigma\right) \cap C\left((0,T)\,; H^{s+2}_\sigma\right)$;
2. u satisfies (NS) with some distribution p such that $\nabla p \in C\left((0,T)\,; H^s\right)$.

Since $s > n/2 - 1$, $H^{s+2} \subset \mathrm{BMO}$, and hence by the definition of the strong solution we have $u \in C((0,T); \mathrm{BMO})$.

Two celebrated theorems

Theorem (Fefferman-Stein (1972))

The dual of $\mathcal{H}^1(\mathbb{R}^n)$ *is* $\mathrm{BMO}(\mathbb{R}^n)$.

They observed

$$\left| \int_{\mathbb{R}^n} fg dx \right| \leq c \, \|f\|_{\mathrm{BMO}} \, \|g\|_{\mathcal{H}^1}$$

for $f \in \mathrm{BMO}$ and $g \in \mathcal{H}^1_a$ so that the integral is well-defined. Here $\mathcal{H}^1_a$ is the space of all g that are bounded and have compact support with $\int g dx = 0$.

See Stein's monograph (1993).

Two celebrated theorems

Theorem (Coifman-Lions-Meyers-Semes (1993))

Let E, B *be vector fields on* $\mathbb{R}^n$ *satisfying* $E \in L^p$ *and* $B \in L^{p'}$ *with* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and*

$$\operatorname{div} E = 0, \quad \operatorname{curl} B = 0 \quad \textit{i.e.,} \quad \partial_j B^i = \partial_i B^j \quad \textit{in } \mathcal{D}'.$$

Then $E \cdot B \in \mathcal{H}^1$ *and there exists a constant* $C > 0$ *such that*

$$\|E \cdot B\|_{\mathcal{H}^1} \leq C \|E\|_{L^p} \|B\|_{L^{p'}}$$

holds.

Two celebrated theorems

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holds.

- First method: maximal function estimates (see original paper or Giaquinta-Martinazzi (2005))
- Second method: Coifman-Rosenberg-Weiss commutator estimates + $\mathrm{VMO}^* = \mathcal{H}^1$. (see original paper or [LPPW] below for idea)
- Recently, there is a multi-parameter generalization given by Lacey-Petermichl-Piper-Wick (2010).

Interpolation of the space BMO

Let $[X_0, X_1]_\theta$ denote the complex interpolation space between X_0 and X_1. (See Lunardi or Bergh-Löfström).

Theorem (Janson-Jones (1982))

Let $0 < p_0 < \infty$ *and* $0 < \theta < 1$. *Then*

$$[\mathcal{H}^{p_0}, L^\infty]_\theta = [\mathcal{H}^{p_0}, \mathrm{BMO}]_\theta = \mathcal{H}^p,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0}.$$

In particular, we can interpolate L^2 and BMO. So

$$L^2 \cap \mathrm{BMO} \subset L^r$$

for any $2 < r < \infty$.

Lemma for convection term

Recall that if $u \in L^r(0,T;L^q)$ is a weak solution of (NS) where (r,q) satisfies (Se):

$$\frac{2}{r} + \frac{n}{q} = 1, \quad n < q \leq \infty, \quad \text{and } 2 \leq r < \infty, \tag{Se}$$

then the energy equality holds:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \, d\tau = \|u_0\|_2^2, \quad 0 < t < T. \tag{6}$$

We will show that the condition $u \in L^2(0,T;\mathrm{BMO})$ gurantees the energy equality.

Lemma for convection term

Lemma

Let $w \in L^\infty\left(0,T;L^2_\sigma\right) \cap L^2\left(0,T;H^1_\sigma\right)$ *and* $u \in L^2\left(0,T;H^1_\sigma \cap \mathrm{BMO}\right)$*. Then we have*

$$\int_0^T \left(\left(w\cdot\nabla\right)u,u\right)d\tau = 0. \tag{7}$$

Proof Fix $1 \le k \le n$. Recall that

$$[(w\cdot\nabla)u]^k = w^i\partial_i u^k.$$

Set $E = w$ and $B = \nabla u^k$. Then $\operatorname{div} E = 0$ and $\operatorname{curl} B = 0$ since

$$\partial_i B^j = \partial_i\partial_j u^k = \partial_j\partial_i u^k = \partial_j B^i \quad \text{in } \mathcal{D}.$$

Hence by the div-curl estimates, $(w\cdot\nabla)u \in \mathcal{H}^1$ and

$$\|(w\cdot\nabla)u\|_{\mathcal{H}^1} \le C\,\|w\|_{L^2}\,\|\nabla u\|_{L^2}.$$

Lemma for convection term

Since $\left(\mathcal{H}^1\right)^* = \mathrm{BMO}$ and $u \in \mathrm{BMO}$, we have

$$\begin{aligned} &\int_0^T (w \cdot \nabla u, u)\, d\tau \qquad (8) \\ &\le c \int_0^T \|w \cdot \nabla u\|_{\mathcal{H}^1} \|u\|_{\mathrm{BMO}}\, d\tau \\ &\le c \sup_{0<\tau<T} \|w(\tau)\|_{L^2} \int_0^T \|\nabla u(\tau)\|_{L^2} \|u(\tau)\|_{\mathrm{BMO}}\, d\tau \\ &\le c \sup_{0<\tau<T} \|w(\tau)\|_{L^2} \left(\int_0^T \|\nabla u(\tau)\|_{L^2}^2\, d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|u(\tau)\|_{\mathrm{BMO}}^2\, d\tau\right)^{\frac{1}{2}} < \infty, \end{aligned}$$

which shows that $\int_0^T ((w \cdot \nabla) u, u)\, d\tau$ is well-defined.

Lemma for convection term

Let $\rho \in C_0^\infty(\mathbb{R}^1)$ with $\operatorname{supp} \rho \subset (-1,1)$ such that $\rho(\tau) = \rho(-\tau)$, $\rho(\tau) \geq 0$, and $\int_{-\infty}^\infty \rho(\tau) = 1$. For $h > 0$, set $\rho_h(\tau) = h^{-1}\rho(h^{-1}\tau)$ and define u_h by

$$u_h(\tau) = \int_0^T \rho_h(\tau - \mu)\, u(\mu)\, d\mu, \quad 0 \leq \tau \leq T. \tag{9}$$

Since $u \in L^2(0,T; H^1_\sigma \cap \mathrm{BMO})$, $u_h \in H^1(0,T; H^1_\sigma \cap \mathrm{BMO})$ and

$$u_h \to u \quad \text{in } L^2(0,T; H^1_\sigma \cap \mathrm{BMO}) \quad \text{as } h \to 0.$$

For such u_h, we claim that

$$\int_0^T ((w\cdot\nabla)u, u_h)\, d\tau = -\int_0^T ((w\cdot\nabla)u_h, u)\, d\tau. \tag{10}$$

If the identity holds, from (8) we have

$$\begin{aligned}&\left|\int_0^T ((w\cdot\nabla)u, u_h)\, d\tau - \int_0^T ((w\cdot\nabla)u, u)\, d\tau\right| \\ &\leq C \sup_{0<\tau<T} \|w(\tau)\|_{L^2} \left(\int_0^T \|\nabla u\|_{L^2}^2\, d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|u_h - u\|_{\mathrm{BMO}}^2\, d\tau\right)^{\frac{1}{2}}\end{aligned}$$

Lemma for convection term

and

$$\left| \int_0^T ((w \cdot \nabla) u_h, u) \, d\tau - \int_0^T ((w \cdot \nabla) u, u) \, d\tau \right|$$

$$\leq C \sup_{0<\tau<T} \|w(\tau)\|_{L^2} \left(\int_0^T \|\nabla u_h - \nabla u\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_0^T \|u\|_{\mathrm{BMO}}^2 \, d\tau \right)^{\frac{1}{2}} .$$

Now since $u_h \to u$ in $L^2 \left(0, T; H^1_\sigma \cap \mathrm{BMO}\right)$ as $h \to 0$, by (10), we obtain

$$\begin{aligned} \int_0^T ((w \cdot \nabla) u, u) \, d\tau &= \lim_{h\to 0} \int_0^T ((w \cdot \nabla) u, u_h) \, d\tau \\ &= - \lim_{h\to 0} \int_0^T ((w \cdot \nabla) u_h, u) \, d\tau \\ &= - \int_0^T ((w \cdot \nabla) u, u) \, d\tau. \end{aligned}$$

This proves $\int_0^T (w \cdot \nabla u, u) \, d\tau = 0$.
Hence it suffices to show that the identity (10) holds. By Janson-Jones' interpolation theorem, we have $L^2 \cap \mathrm{BMO} \subset L^n$. So for each fixed $h > 0$, $u_h \in H^1 \left(0, T; H^1_\sigma \cap L^n\right)$.

Lemma for convection term

Lemma (Masuda, Proposition 1 and Lemma 2.2 (1984))

1. $C_{0,\sigma}^{\infty}$ *is dense in* $H_{\sigma}^{1} \cap L^{n}$.[1]
2. *Let* X_0 *be a dense subset of a Banach space* X. *Then any function* $\Phi \in H^1((s,t);X)$ *can be approximated by a sequence* $\{\Phi_k\}$ *in the topology of* $H^1((s,t);X)$ *such that each* Φ_k *has the form*

$$\Phi_k(\tau) = \sum_{\textit{finite}} \lambda_j(\tau)\phi_j,$$

where λ_j *is some* C^∞*-function on* $\mathbb{R}$ *and* ϕ_j *is some element of* X_0.

By this lemma, there is a sequence $\{u_h^k\}_{k=1}^{\infty}$ of functions having the form

$$u_h^k(t) = \sum_{\text{finite}} \lambda_j^{(k)}(t)\,\phi_j^{(k)} \quad \text{with } \lambda_j^{(k)} \in C^\infty([0,T]),\, \phi_j^{(k)} \in C_{0,\sigma}^{\infty} \tag{11}$$

such that

$$u_h^k \to u_h \quad \text{in } H^1\left(0,T; H_\sigma^1 \cap L^n\right)$$

as $k \to \infty$.

[1] In general, if $1 \le p < \infty$ and Ω is a bounded domain or exterior domain in $\mathbb{R}^n$, then $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $H_{0,\sigma}^{1} \cap L^p(\Omega)$.

Lemma for convection term

Since u_h^k is a finite linear combination of smooth functions, we can perform the integration by parts to get

$$\int_0^T \left((w \cdot \nabla) u, u_h^k \right) d\tau = -\int_0^T \left((w \cdot \nabla) u_h^k, u \right) d\tau.$$

Now Hölder inequality and Sobolev inequality give

$$\begin{aligned} &\left| \int_0^T \left((w \cdot \nabla) u, u_h^k \right) d\tau - \int_0^T \left((w \cdot \nabla) u, u_h \right) d\tau \right| \qquad (12) \\ &\leq \int_0^T \|w\|_{L^{\frac{2n}{n-2}}} \|\nabla u\|_{L^2} \left\| u_h^k - u_h \right\|_{L^n} d\tau \\ &\leq C \int_0^T \|\nabla w\|_{L^2} \|\nabla u\|_{L^2} \left\| u_h^k - u_h \right\|_{L^n} d\tau \\ &\leq C \sup_{0<\tau<T} \left\| u_h^k(\tau) - u_h(\tau) \right\|_{L^n} \left(\int_0^T \|\nabla w\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla u\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\operatorname{div} w = 0$ in Ω, applying the div-curl estimate, $\mathcal{H}^1$-BMO duality, and Hölder's inequality, we have

$$\begin{aligned}
&\left| \int_0^T \left((w \cdot \nabla) u_h^k, u \right) d\tau - \int_0^T ((w \cdot \nabla) u_h, u) \, d\tau \right| \\
&\leq c \int_0^T \left\| w \cdot \left(\nabla u_h^k - \nabla u_h \right) \right\|_{\mathcal{H}^1} \|u\|_{\mathrm{BMO}} \, d\tau \\
&\leq c \int_0^T \|w\|_{L^2} \left\| \nabla u_h^k - \nabla u_h \right\|_{L^2} \|u\|_{\mathrm{BMO}} \, d\tau \\
&\leq c \sup_{0<\tau<T} \|w(\tau)\|_{L^2} \left(\int_0^T \left\| \nabla u_h^k - \nabla u_h \right\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^T \|u\|_{\mathrm{BMO}}^2 \, d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

Now letting $k \to \infty$, this proves

$$\int_0^T ((w \cdot \nabla) u, u_h) \, d\tau = - \int_0^T ((w \cdot \nabla) u_h, u) \, d\tau. \qquad \square$$

Energy equality

As a consequence of the previous lemma, we obtain the energy equality for (NS).

Lemma

Let $u_0 \in L^2_\sigma$. *Suppose that* u *is a weak solution of* (NS) *on* $(0,T)$ *satisfying* $u \in L^2(0,T;\mathrm{BMO})$. *Then* u *satisfies the energy equality*

$$\|u(t)\|^2_{L^2} + 2\int_s^t \|\nabla u\|^2_{L^2}\, d\tau = \|u(s)\|^2_{L^2} \quad \textit{for all } 0 \le s \le t < T.$$

Proof By Janson-Jones' interpoaltion theorem, we have

$$u \in L^2(0,T;L^2 \cap \mathrm{BMO}) \subset L^2(0,T;L^n).$$

Let $\rho_h, h > 0$ be the same function as in the proof of previous Lemma. Choose

$$\Phi(\tau) = u_h(\tau) = \int_s^t \rho_h(\tau - \mu)\, u(\mu)\, d\mu.$$

Then $u_h \in H^1\left((s,t); H^1_\sigma \cap L^n\right)$.

By the symmetry of ρ_h, we have

$$\begin{aligned}\int_s^t \big(u(\tau), (u_h)'(\tau)\big)\, d\tau &= \int_s^t \int_s^t \rho_h'(\tau-\mu)\,(u(\tau), u(\mu))\, d\mu d\tau \\ &= -\int_s^t \int_s^t \rho_h'(\tau-\mu)\,(u(\tau), u(\mu))\, d\mu d\tau \\ &= -\int_s^t \big(u(\tau), (u_h)'(\tau)\big)\, d\tau.\end{aligned}$$

So

$$\int_s^t \big(u(\tau), (u_h)'(\tau)\big)\, d\tau = 0.$$

Since u is a weak solution of (NS), u is weakly continuous on $[0, T]$ in the weak topology of L^2_σ. By definition of ρ_h, we have

$$-\left(u(t), u_h(t)\right) \to -\frac{1}{2}\|u(t)\|_{L^2}^2, \quad \left(u(s), u_h(s)\right) \to \frac{1}{2}\|u(s)\|_{L^2}^2$$

as $h \to 0$.

Energy equality

Since $u \in L^2\left(0, T; H^1_\sigma\right)$, we see that

$$\left|\int_s^t \left(\nabla u, \nabla u_h\right) d\tau - \int_s^t \left(\nabla u, \nabla u\right) d\tau\right|$$
$$\leq \int_s^t \|\nabla u\|_{L^2} \|\nabla u_h - \nabla u\|_{L^2} \, d\tau$$
$$\leq \left(\int_s^t \|\nabla u\|_{L^2}^2 \, d\tau\right)^{\frac{1}{2}} \left(\int_s^t \|\nabla u_h - \nabla u\|_{L^2}^2 \, d\tau\right)^{\frac{1}{2}}.$$

Now letting $h \to 0$, we see that

$$\lim_{h\to 0} \int_s^t \left(\nabla u, \nabla u_h\right) d\tau = \frac{1}{2} \int_s^t \|\nabla u\|_{L^2}^2 \, d\tau.$$

Since $u \in L^2\left(0, T; H^1_\sigma\right) \cap L^2(0, T; \mathrm{BMO})$, div-curl estimate gives $\left(u \cdot \nabla\right) u \in \mathcal{H}^1$ with

$$\left\|\left(u \cdot \nabla\right) u\right\|_{\mathcal{H}^1} \leq c \left\|u\right\|_{L^2} \left\|\nabla u\right\|_{L^2}.$$

Energy equality

By $\mathcal{H}^1$-BMO duality, we have

$$\begin{aligned}
&\left|\int_s^t ((u\cdot\nabla)\,u, u_h)\,d\tau - \int_s^t ((u\cdot\nabla)\,u, u)\,d\tau\right| \\
&\leq C\int_s^t \|u\|_{L^2}\,\|\nabla u\|_{L^2}\,\|u_h-u\|_{\mathrm{BMO}}\,d\tau \\
&\leq C\sup_{s<\tau<t}\|u(\tau)\|_{L^2}\left(\int_s^t \|\nabla u\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_s^t \|u_h-u\|_{\mathrm{BMO}}^2\,d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

Now letting $h\to 0$, we get

$$\lim_{h\to 0}\int_s^t ((u\cdot\nabla)\,u, u_h)\,d\tau = \int_s^t ((u\cdot\nabla)\,u, u)\,d\tau.$$

Now take $\Phi = u_h$ in (NS) and then let $h\to 0$. Then from the above analysis, we have

$$\int_s^t \|\nabla u\|_{L^2}^2\,d\tau = -\frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|u(s)\|_{L^2}^2$$

for all $0\leq s\leq t<T$. This completes the proof.

Proof of regularity criteria

We are ready to prove the following theorem:

Theorem (Kozono-Taniuchi (2000))

1. *Let* $u_0 \in L^2_\sigma$ *and let* u, v *be two weak solutions of* (NS) *on* $(0, T)$*. Suppose that*
$$u \in L^2\left(0, T; \mathrm{BMO}\right) \tag{13}$$
and that v *satisfies the energy inequality*
$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(\tau)\|_2^2 \, d\tau \le \|u_0\|_2^2, \quad 0 < t < T. \tag{14}$$
Then we have $u = v$ *on* $[0, T]$*.*
2. *Let* $u_0 \in L^2_\sigma$ *and let* u *be a weak solution with the additional property* (13)*. Then for every* $0 < \varepsilon < T$*,* u *is actually a strong solution of* (NS) *in the class* $CL_s(\varepsilon, T)$ *for* $s > \frac{n}{2} - 1$*.*

Proof (1) First, we show the uniqueness. Suppose u and v are solutions of (NS) with energy inequality. Set $s = 0$ in the definition of weak solution of (NS). Take two test functions u_h and v_h^k in the same way as in (11). Then $v_h^k \to v$ in $L^2\left(0, T; H_\sigma^1\right)$ as $k \to \infty$, $h \to 0$.
Since $u \in L^2\left(0, T; \mathrm{BMO}\right)$, we have by integration by parts

$$\int_0^t \left((u \cdot \nabla)\, u, v_h^k\right) d\tau = -\int_0^t \left((u \cdot \nabla)\, v_h^k, u\right) d\tau \to -\int_0^t ((u \cdot \nabla)\, v, u)\, d\tau$$

as $k \to \infty$ and $h \to 0$.
Similarly, we have

$$\int_0^t ((v \cdot \nabla)\, v, u_h)\, d\tau \to \int_0^t ((v \cdot \nabla)\, v, u)\, d\tau$$

as $h \to 0$.

Hence, we have the identity

$$\int_0^t \{2(\nabla u, \nabla v) + ((v \cdot \nabla v), u) - ((u \cdot \nabla) v, u)\} \, d\tau \\ = -(u(t), v(t)) + \|u_0\|_{L^2}^2 . \tag{15}$$

Since $u \in L^2(0, T; \mathrm{BMO})$, u satisfies the energy equality

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 . \tag{16}$$

Set $w = u - v$. Multiply -2 to (15). Then add this with the energy equality on u and energy inequality of v. Then by calculation, we get

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w\|_{L^2}^2 \, d\tau \leq 2\int_0^t ((w \cdot \nabla) v, u) \, d\tau.$$

Proof of regularity criteria

Then applying the div-curl estimate, $\mathcal{H}^1$-BMO duality and Cauchy's inequality and using the Lemma, we get

$$\begin{aligned}
\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w\|_{L^2}^2 \, d\tau &\le 2\int_0^t ((w\cdot\nabla)v, u)\, d\tau \\
&= 2\int_0^t ((w\cdot\nabla)w, u)\, d\tau \\
&\le C\int_0^t \|w\cdot\nabla w\|_{\mathcal{H}^1} \|u\|_{\mathrm{BMO}}\, d\tau \\
&\le C\int_0^t \|w\|_{L^2} \|\nabla w\|_{L^2} \|u\|_{\mathrm{BMO}}\, d\tau \\
&\le \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau + C\int_0^t \|w\|_{L^2}^2 \|u\|_{\mathrm{BMO}}^2 \, d\tau.
\end{aligned}$$

Hence

$$\|w(t)\|_{L^2}^2 \le C\int_0^t \|w\|_{L^2}^2 \|u\|_{\mathrm{BMO}}^2 \, d\tau, \quad 0 \le t < T.$$

Since $u \in L^2(0,T;\mathrm{BMO})$, the Gronwall inequality yields

$$\|w(t)\|_{L^2}^2 = 0, \quad 0 \le t < T.$$

So we get the desired uniqueness result.

Proof of regularity criteria

(2) Next, we show the regularity. Since $u \in L^2\left(0,T;H^1_\sigma \cap \mathrm{BMO}\right)$, for every $0<\varepsilon<T$, there is $0<\delta<\varepsilon$ such that $u(\delta) \subset H^1_\sigma \cap \mathrm{BMO} \subset L^2_\sigma \cap L^r_\sigma$ for $n<r<\infty$. The last inclusion follows from Janson-Jones' theorem. Hence by the local existence of strong solution of (NS), there are $T_* > \delta$ and a unique strong solution $\tilde{u}$ on $[\delta, T_*)$ with $\tilde{u}\mid_{t=\delta} = u(\delta)$ such that

$$\tilde{u} \in C\left([\delta,T_*);H^1_\sigma \cap L^r_\sigma\right) \cap C^1\left((\delta,T_*):H^{s+2}\right) \quad \text{for } s > \frac{n}{2}-1.$$

Since $u \in L^2(0,T;\mathrm{BMO})$, u satisfies the energy equality

$$\|u(t)\|^2_{L^2} + 2\int_\delta^t \|\nabla u\|^2_{L^2}\,d\tau = \|u(\delta)\|^2_{L^2}, \quad \delta \le t < T.$$

Since $\tilde{u} \in L^{r'}(0,T;L^r)$, where $n<r<\infty$ and r' satisfy

$$\frac{n}{r}+\frac{2}{r'} \le 1,$$

by Serrin-Masuda's criterion, $u \equiv \tilde{u}$ on $[\delta, T_*)$. So we can regard u as a strong solution in the class $CL_s(\delta', T_*)$ for $\delta<\delta'<\varepsilon$.

Proof of regularity criteria

We claim that $T_* = T$. If not, then there exists $T_0 < T$ such that u is a strong solution in the class $CL_s(\delta', T_0)$ but cannot be continued in the class $CL_s\left(\delta', \tilde{T}\right)$ for $\tilde{T} > T_0$. Note that we have $u \in L^2(0, T; \mathrm{BMO})$. So

$$\int_{\delta'}^{T_0} \|u\|_{\mathrm{BMO}}^2 \, d\tau \le \int_0^T \|u\|_{\mathrm{BMO}}^2 \, d\tau < \infty.$$

But this contradicts the blow-up result. So $T_* = T$. This completes the proof of the regularity assertion of Theorem.

Corollary (Blow-up result)

Let u be a strong solution of (NS) *in the class $CL_s(0, T)$ for $s > \frac{n}{2} - 1$. Suppose that T is maximal, i.e., u cannot be continued in the class $CL_s(0, T')$ for any $T' > T$. Then*

$$\int_\varepsilon^T \|u(t)\|_{\mathrm{BMO}}^2 \, dt = \infty \quad \textit{for any } 0 < \varepsilon < T.$$

In particular, we have

$$\limsup_{t \to T-} \|u(t)\|_{\mathrm{BMO}} = \infty.$$

□

The following bilinear estimates is crucial to prove the regularity criteria in terms of vorticity and deformation tensor.

Lemma

Let $1 < r < \infty$*.*

(1) *There exists a constant* $C = C(n, r)$ *such that*

$$\|f \cdot g\|_{L^r} \le C \left(\|f\|_{L^r} \|g\|_{\mathrm{BMO}} + \|f\|_{\mathrm{BMO}} \|g\|_{L^r} \right)$$

for all $f, g \in L^r \cap \mathrm{BMO}$*.*

(2) *There exists a constant* $C = C(n, r)$ *such that*

$$\|f \cdot \nabla g\|_{L^r} \le C \left(\|f\|_{L^r} \left\| (-\triangle)^{\frac{1}{2}} g \right\|_{\mathrm{BMO}} + \left\| (-\triangle)^{\frac{1}{2}} f \right\|_{\mathrm{BMO}} \|g\|_{L^r} \right)$$

for all $f, g \in W^{1,r}$ *with* $\nabla f, \nabla g \in \mathrm{BMO}$*.*

Bilinear estimates in BMO

The following bilinear estimates is crucial to prove the regularity criteria in terms of vorticity and deformation tensor.

Lemma

Let $1 < r < \infty$.

(3) *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ *be multi-indices with* $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 1$ *and* $|\beta| = \beta_1 + \cdots + \beta_n \geq 1$. *Then there exists a constant* $C = C(n, r, \alpha, \beta)$ *such that*

$$\left\| D^\alpha f \cdot D^\beta g \right\|_{L^r} \leq C \left(\|f\|_{\mathrm{BMO}} \left\| (-\triangle)^{\frac{|\alpha|+|\beta|}{2}} g \right\|_{L^r} + \left\| (-\triangle)^{\frac{|\alpha|+|\beta|}{2}} f \right\|_{L^r} \|g\|_{\mathrm{BMO}} \right)$$

for all $f, g \in \mathrm{BMO} \cap W^{|\alpha|+|\beta|, r}$.

Proof requires the theory of Coifman-Meyer theory of bilinear operators.

Recall the following classical theorem on the theory of singular integral:

Theorem (Mikhlin's multiplier theorem)

Let $m(\xi)$ be a complex-valued bounded funciton on $\mathbb{R}^n \setminus \{0\}$ that satisfies

$$|\partial_\xi^\alpha m(\xi)| \leq A|\xi|^{-|\alpha|}$$

for all multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$. Define

$$Tf(x) = c\int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi, \quad f \in \mathcal{S}.$$

Then T is a bounded linear operator from L^p to itself for any $1 < p < \infty$

Coifman-Meyer theory of bilinear operators

Theorem (Coifman-Meyer)

Let $\sigma = \sigma(\xi, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\})$ *satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \quad (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$$

for all multi-indices α, β *with* $C = C(\alpha, \beta)$. *Suppose that*

$$\sigma(\xi, 0) = 0.$$

Then the bilinear operator $\sigma(D)(\cdot,\cdot)$ *defined by*

$$\sigma(D)(f,g)(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix\cdot(\xi+\eta)} \sigma(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta, \quad x \in \mathbb{R}^n \tag{17}$$

satisfies

$$\|\sigma(D)(f,g)\|_{L^2} \le C \|f\|_{L^2} \|g\|_{\mathrm{BMO}}$$

with $C = C(n)$.

* Authors made the wrong citation.

Remark

1. The expression (17) has make sense for f and g in the Wiener algebra. Then $\sigma(D)$ can extend to a bicontinuous operator from $L^2 \times \mathrm{BMO}$ to L^2.
2. Let $1 < r < \infty$ and $g \in \mathrm{BMO}$. Define $T(f) = \sigma(D)(f, g)$. Then T is a Calderón-Zygmund operator. So

$$\|\sigma(D)(f,g)\|_{L^r} \leq C \|f\|_{L^r} \|g\|_{\mathrm{BMO}}.$$

3. The proof is quite difficult. The proof uses $T(1)$ theorems with some analysis on 'strict convergence in BMO'.

Proof We only prove the case (i). The proof of rest parts are essentially same. It suffices to prove when $f, g \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class. Let $\Phi_1 \in C^\infty([0,\infty))$ such that $\operatorname{supp}\Phi_1 \subset [0,1)$, $0 \le \Phi_1 \le 1$, $\Phi_1(t) = 1$ for $0 \le t \le 1/2$ and $\Phi_2 = 1 - \Phi_1$. For $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$,

$$\sigma_j(\xi,\eta) = \Phi_j\left(\frac{|\xi|}{|\eta|}\right) \quad \text{for } j = 1,2.$$

For $\eta \neq 0$, $\sigma_2(0,\eta)$ is well-defined and $\sigma_2(0,\eta) = 0$. Fix $\xi \neq 0$. Since $\operatorname{supp}\Phi_1 \subset [0,1)$, for any $\eta \neq 0$ with $|\eta| < |\xi|$, $\sigma_1(\xi,\eta) = 0$. Hence, for each $\xi \neq 0$, $(\xi,0)$ is a removable singularity of σ_1 and $\sigma_1(\xi,0) = 0$.

Recall

$$\frac{\partial}{\partial\xi_i}(|\xi|) = \frac{\xi_i}{|\xi|}, \quad \frac{\partial}{\partial\eta_i}\left(\frac{1}{|\eta|}\right) = -\frac{1}{|\eta|^2}\frac{\eta_i}{|\eta|}.$$

Note that

$$\partial_{\xi_i}\sigma_1(\xi,\eta) = \partial_{\xi_i}\left(\Phi_1\left(\frac{|\xi|}{|\eta|}\right)\right) = \Phi_1'\left(\frac{|\xi|}{|\eta|}\right)\frac{1}{|\eta|}\frac{\xi_i}{|\xi|},$$

$$\partial_{\eta_i}\sigma_1(\xi,\eta) = \partial_{\eta_i}\left(\Phi_1\left(\frac{|\xi|}{|\eta|}\right)\right) = \Phi_1'\left(\frac{|\xi|}{|\eta|}\right)\left(-\frac{1}{|\eta|^2}\frac{\eta_i}{|\eta|}\right)|\xi|.$$

Proof of bilinear estimates in BMO

Since $\operatorname{supp}\Phi_j' \subset [1/2, 1)$ for $j = 1, 2$,

$$\frac{|\xi|}{|\eta|} \in \operatorname{supp}\Phi_j' \iff \frac{1}{2} \leq \frac{|\xi|}{|\eta|} < 1.$$

So

$$\left|\partial_{\xi_i}\sigma_1(\xi,\eta)\right| \leq \frac{c}{|\eta|} \leq \frac{c}{|\xi|},$$

$$\left|\partial_{\eta_i}\sigma_1(\xi,\eta)\right| \leq c\frac{|\xi|}{|\eta|^2} \leq \frac{c}{|\eta|}.$$

Hence we get

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)\right| \leq \frac{c}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}} \quad \text{for } (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$$

for all multi-indices α, β with $C = C(\alpha, \beta)$.

Proof of bilinear estimates in BMO

Write

$$f(x)\,g(x) = c\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{ix\cdot(\xi+\eta)}\hat{f}(\xi)\,\hat{g}(\eta)\,d\xi d\eta$$
$$= c\left(\sigma_1(D)(f,g)(x) + \sigma_2(D)(f,g)(x)\right).$$

Since σ_1, σ_2 satisfy the hypothesis of Coifman-Meyer theorem, we have

$$\|fg\|_{L^r} \le c\,\|\sigma_1(D)(f,g)\|_{L^r} + c\,\|\sigma_2(D)(f,g)\|_{L^r}$$
$$\le c\,\|f\|_{L^r}\|g\|_{\mathrm{BMO}} + c\,\|f\|_{\mathrm{BMO}}\|g\|_{L^r}.$$

This completes the proof. □

Lemma for convection term

For a vector field $u : \mathbb{R}^n \to \mathbb{R}^n$, we write $u = (u^1, \dots, u^n)$ and

$$\operatorname{curl} u = \left(D_j u^k - D_k u^j\right)_{1 \le j,k \le n}, \quad \operatorname{Def} u = \left(D_j u^k + D_k u^j\right)_{1 \le j,k \le n}.$$

Lemma

Let $w, u \in L^\infty\left(0, T; L^2_\sigma\right) \cap L^2\left(0, T; H^1_\sigma\right)$. *Suppose that either*

$$\operatorname{curl} w, \operatorname{curl} u \in L^1\left(0, T; \mathrm{BMO}\right)$$

or

$$\operatorname{Def} w, \operatorname{Def} u \in L^1\left(0, T; \mathrm{BMO}\right)$$

holds. Then we have

$$\int_0^T \left(\left(w \cdot \nabla\right) u, u\right) d\tau = 0. \tag{18}$$

Lemma for convection term

Lemma (Biot-Savart law)

Let $1 < q < \infty$ *and* $u \in L^{1,q}_\sigma$. *Then we have*

$$\frac{\partial u}{\partial x_j} = R_j \left(R \times \omega\right), \quad j = 1, \ldots, n, \quad \textit{where } \omega = \operatorname{curl} u;$$

$$\frac{\partial u^l}{\partial x_j} = R_j \left(\sum_{k=1}^{n} R_k \operatorname{Def} u_{kl}\right), \quad j, l = 1, \ldots, n,$$

where

$$(\operatorname{curl} u)_{jk} = \partial_j u^k - \partial_k u^j \quad \textit{and} \quad \operatorname{Def} u_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.$$

Here $R = (R_1, \ldots, R_n)$, $R_j = \frac{\partial}{\partial x_j} (-\triangle)^{\frac{1}{2}}$ *denote the Riesz transforms.*

Lemma for convection term

Proof Here we only prove

$$\frac{\partial u}{\partial x_j} = R_j\left(R \times \omega\right), \quad j = 1, \ldots, n, \quad \text{where } \omega = \operatorname{curl} u$$

for $u \in C^{\infty}_{0,\sigma}$.
Fix $1 \leq i \leq n$. By linearility of Riesz transform and definition of vorticity, we have

$$[R_j(R \times \omega)]^i = R_j(R_k(\partial_k u^i - \partial_i u^k)) = R_j R_k(\partial_k u^i) - R_j R_k(\partial_i u^k).$$

Observe that

$$R_j R_k(\partial_i u^k) = R_i R_j(\partial_k u^k).$$

Indeed,

$$\begin{aligned}(R_j R_k(\partial_i u^k))^\wedge(\xi) &= \frac{i\xi_j}{|\xi|}\frac{i\xi_k}{|\xi|} i\xi_i \widehat{u^k}(\xi) \\ &= \frac{i\xi_i}{|\xi|}\frac{i\xi_j}{|\xi|} i\xi_k \widehat{u^k}(\xi) \\ &= (R_i R_j(\partial_k u^k))^\wedge(\xi)\end{aligned}$$

and taking the inverse Fourier transform, the identity follows.

So

$$[R_j(R\times\omega)]^i = R_j(R_k(\partial_k u^i - \partial_i u^k))$$
$$= R_jR_k(\partial_k u^i) - R_jR_k(\partial_i u^k)$$
$$= R_jR_k(\partial_k u^i) - R_iR_j(\partial_k u^k).$$

Since $\operatorname{div} u = 0$ in $\mathbb{R}^n$, we have

$$[R_j(R\times\omega)]^i = R_jR_k(\partial_k u^i) = R_kR_k(\partial_j u^i) = \frac{\partial u^i}{\partial x_j}.$$

Here we used

$$\sum_{k=1}^n R_k^2 = I.$$

This proves the Biot-Savart Law. □

Proof of Lemma By the Biot-Savart Law, we have

$$\frac{\partial u}{\partial x_j} = R_j \left(R \times \omega\right), \quad , j = 1, \ldots, n, \quad \text{where } \omega = \operatorname{curl} u;$$

$$\frac{\partial u^l}{\partial x_j} = R_j \left(\sum_{k=1}^{n} R_k \operatorname{Def} u_{kl} \right), \quad j, l = 1, \ldots, n,$$

where

$$\operatorname{Def} u_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.$$

Here $R = (R_1, \ldots, R_n)$, $R_j = \frac{\partial}{\partial x_j} \left(-\triangle\right)^{\frac{1}{2}}$ denote the Riesz transforms. Since $R : \mathrm{BMO} \to \mathrm{BMO}$, $\nabla u, \nabla w \in L^1\left(0, T; \mathrm{BMO}\right)$ if $\operatorname{curl} u, \operatorname{curl} w \in L^1\left(0, T; \mathrm{BMO}\right)$ or $\operatorname{Def} u, \operatorname{Def} w \in L^1\left(0, T; \mathrm{BMO}\right)$.

Lemma for convection term

By bilinear estimates in BMO, we have

$$\begin{aligned}
&\int_0^T \|(w \cdot \nabla)u\|_{L^2} \, d\tau \\
&\lesssim \int_0^T \|w\|_{L^2} \|\nabla u\|_{\mathrm{BMO}} + \|\nabla w\|_{\mathrm{BMO}} \|u\|_{L^2} \, d\tau \\
&\lesssim \|w\|_{L^\infty(0,T;L^2)} \|\nabla u\|_{L^1(0,T;\mathrm{BMO})} + \|u\|_{L^\infty(0,T;L^2)} \|\nabla w\|_{L^1(0,T;\mathrm{BMO})} < \infty.
\end{aligned}$$

So $(w \cdot \nabla)u \in L^1\left(0, T; L^2\right)$. Since $u \in L^\infty\left(0, T; L^2\right)$, we have

$$\int_0^T \left((w \cdot \nabla)\, u, u\right) d\tau < \infty.$$

Hence, the integral is well-defined.

Lemma for convection term

Let $\rho \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \rho \subset (-1,1)$ such that $\rho(\tau) = \rho(-\tau)$, $\rho(\tau) \geq 0$ and $\int_{\mathbb{R}} \rho d\tau = 1$. For $h > 0$, we set $\rho_h(\tau) = h^{-1} \rho\left(h^{-1}\tau\right)$ and define

$$u_h(\tau) = \int_0^T \rho_h(\tau - \mu) u(\mu) d\mu, \quad 0 \leq t \leq T.$$

Assume in a moment that

$$\int_0^T ((w \cdot \nabla) u, u_h) d\tau = -\int_0^T ((w \cdot \nabla) u_h, u) d\tau.$$

Since $u_h \to u$ weakly-star in $L^\infty\left(0,T;L^2\right)$,

$$\lim_{h \to 0} \int_0^T ((w \cdot \nabla) u, u_h) d\tau = \int_0^T ((w \cdot \nabla) u, u) d\tau$$

because $(w \cdot \nabla) u \in L^1\left(0,T;L^2\right)$.

Lemma for convection term

By bilinear estimates in BMO, we have

$$\begin{aligned}
&\left|\int_0^T ((w\cdot\nabla)\,u_h, u)\,d\tau - \int_0^T ((w\cdot\nabla)\,u, u)\,d\tau\right| \\
&= \left|\int_0^T ((w\cdot\nabla)\,(u_h-u)\,, u)\,d\tau\right| \\
&\le \|u\|_{L^\infty(0,T;L^2)}\,\|w\cdot\nabla\,(u_h-u)\|_{L^1(0,T;L^2)} \\
&\le C\,\|u\|_{L^\infty(0,T;L^2)}\,\|w\|_{L^\infty(0,T;L^2)}\,\|\nabla u_h - \nabla u\|_{L^1(0,T;\mathrm{BMO})} \\
&\quad + C\,\|u\|_{L^\infty(0,T;L^2)}\,\|\nabla w\|_{L^1(0,T;\mathrm{BMO})}\,\|u_h - u\|_{L^\infty(0,T;L^2)} \\
&= I_h^{(1)} + I_h^{(2)}.
\end{aligned}$$

So $\nabla u_h \to \nabla u$ in $L^1(0,T;\mathrm{BMO})$ and hence $I_h^{(1)} \to 0$ as $h \to 0$.
Since $u_h \to u$ in $L^2(0,T;L^2)$, there exists a subsequence $\left\{u_{h_j}\right\}$ with $h_j \to 0$ as $j \to \infty$ such that

$$\lim_{j\to\infty}\left\|u_{h_j}(\tau) - u(\tau)\right\|_{L^2} = 0 \quad \text{for almost all } \tau \in (0,T).$$

Since

$$\|\nabla w(\tau)\|_{\mathrm{BMO}} \|u_h(\tau) - u(\tau)\|_{L^2} \le 2\|u\|_{L^\infty(0,T;L^2)} \|\nabla w(\tau)\|_{\mathrm{BMO}}, \quad 0 < \tau < T$$

for all $h > 0$. Since $\nabla w \in L^1(0,T;\mathrm{BMO})$, $I_{h_j}^{(2)} \to 0$ as $j \to \infty$ by dominated convergence theorem. Thus,

$$\lim_{j\to\infty} \int_0^T \Big((w\cdot\nabla) u_{h_j}, u \Big) \, d\tau = \int_0^T ((w\cdot\nabla) u, u) \, d\tau.$$

Hence

$$\int_0^T ((w\cdot\nabla) u, u) \, d\tau = -\int_0^T ((w\cdot\nabla) u, u) \, d\tau,$$

which proves

$$\int_0^T ((w\cdot\nabla) u, u) \, d\tau = 0.$$

It remains to prove the identity

$$\int_0^T ((w\cdot\nabla) u, u_h) \, d\tau = -\int_0^T ((w\cdot\nabla) u_h, u) \, d\tau.$$

Note that $\nabla u \in L^1(0,T;\mathrm{BMO}) \cap L^1\left(0,T;L^2\right) \subset L^1(0,T;L^n)$ by Janson-Jones' interpolation theorem.

Lemma for convection term

Since

$$\|u(\tau)\|_{\mathrm{BMO}} \leq C\|\nabla u(\tau)\|_{L^n} \quad \text{for a.e. } \tau,$$

for some constant C which does not depends on τ, we have $u \in L^1(0,T;\mathrm{BMO})$.
Hence

$$\sup_{0<\tau<T}\|u_h(\tau)\|_{L^n} \leq M_h, \quad \sup_{0<\tau<T}\|\nabla u_h(\tau)\|_{L^n} \leq M_h \tag{19}$$

with a constant M_h depending on h.
Since $C_{0,\sigma}^{\infty}$ is dense in H_{σ}^1, by the Lemma of Masuda, we can choose a sequence $\{u^k\}_{k=1}^{\infty}$ having the form of

$$u_h^k(\tau) = \sum_{\text{finite}} \lambda_j^{(k)}(t)\,\phi_j^{(k)}, \quad \text{with } \lambda_j^{(k)} \in C^{\infty}([0,T]),\, \phi_j^{(k)} \in C_{0,\sigma}^{\infty}$$

such that

$$u^k \to u \quad \text{in } L^2(0,T;H_{\sigma}^1) \quad \text{as } k \to \infty. \tag{20}$$

For such u^k, we have

$$\int_0^T \left((w\cdot\nabla)u^k, u_h\right)d\tau = -\int_0^T \left((w\cdot\nabla)u_h, u^k\right)d\tau.$$

Lemma for convection term

By (19), (20) and the Sobolev inequality, we have

$$\begin{aligned}
&\left|\int_0^T \left((w\cdot\nabla)\, u^k, u_h\right) d\tau - \int_0^T ((w\cdot\nabla)\, u, u_h)\, d\tau\right| \\
&\le \int_0^T \|w\|_{L^{\frac{2n}{n-2}}} \left\|\nabla u^k - \nabla u\right\|_{L^2} \|u_h\|_{L^n}\, d\tau \\
&\le M_h \left(\int_0^T \|\nabla w\|_{L^2}^2\, d\tau\right)^{\frac{1}{2}} \left(\int_0^T \left\|\nabla u^k - \nabla u\right\|_{L^2}^2 d\tau\right)^{\frac{1}{2}} \to 0,
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^T \left((w\cdot\nabla)\, u_h, u^k\right) d\tau - \int_0^T ((w\cdot\nabla)\, u_h, u)\, d\tau\right| \\
&\le \int_0^T \|w\|_{L^2} \|\nabla u_h\|_{L^n} \left\|u^k - u\right\|_{L^{\frac{2n}{n-2}}} d\tau \\
&\le CM_h \|w\|_{L^\infty(0,T;L^2)} \int_0^T \left\|\nabla u^k - \nabla u\right\|_{L^2} d\tau \to 0
\end{aligned}$$

as $k \to \infty$. Thus,

$$\int_0^T ((w\cdot\nabla)\, u, u_h)\, d\tau = -\int_0^T ((w\cdot\nabla)\, u_h, u)\, d\tau. \qquad \square$$

Energy equality

Lemma

Let $u_0 \in L^2_\sigma$. Suppose that u is a weak solution of (NS) *on $(0,T)$ satisfying one of the additional conditions*

1. $\operatorname{curl} u \in L^1(0,T;\mathrm{BMO})$
2. $\operatorname{Def} u \in L^1(0,T;\mathrm{BMO})$.

Then u satisfies the energy equality

$$\|u(t)\|_{L^2}^2 + 2\int_s^t \|\nabla u\|_{L^2}^2\, d\tau = \|u(s)\|_{L^2}^2 \quad \textit{for all } 0 \le s \le t < T.$$

The proof is similar to the case of $u \in L^2(0,T;\mathrm{BMO})$. We omit it.

Following the argument as in the case of $u \in L^2(0,T;\mathrm{BMO})$, we obtain the following theorem.

Theorem

Let $u_0 \in L^2_\sigma$. Suppose that u is a weak solution of (NS) *on $(0,T)$. If either*

$$\operatorname{curl} u \in L^1\left(0,T;\mathrm{BMO}\right)$$

or

$$\operatorname{Def} u \in L^1\left(0,T;\mathrm{BMO}\right)$$

holds, then for every $0 < \varepsilon < T$, u is actually a strong solution of (NS) *in the class $CL_s\left(\varepsilon,T\right)$ for $s > \frac{n}{2} - 1$.*

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See the rest references in the abstract.

Thank you

Thank you for your attentions!